MATH 249 Notes

Benjamin Chen

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1 January 6th

David Wagner

1.1 Enumeration

- Solving counting problems.
  - Bijective / combinatorial
  - Algebraic

Read: New course notes
Chapter 1, beginning of Chapter 2, beginning of Chapter 4
Examples: Fibonacci Numbers

- Initial conditions: $f_0 = 1, f_1 = 1$.
- Recurrence Relation: For $n \geq 2 : f_n = f_{n-1} + f_{n-2}$.

n	0	1	2	3	4	5	6	7	8	9
f_n	1	1	2	3	5	8	13	21	34	55

What is $f_{10^{10^{10}}}$
What is f_n as a function of n?
Define the generating series,

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$$

- Get a formula for $F(x)$
- Use this to get a formula for f_n.

$$
\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} f_n x^n \\
&= f_0 + f_1 x + \sum_{n=2}^{\infty} f_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n \\
&= 1 + x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \\
&= 1 + x + x \sum_{j=1}^{\infty} f_j x^j + x^2 \sum_{k=0}^{\infty} f_k x^k \\
&= 1 + x + (F(x) - f_0) + x^2 (F(x)) \\
&= 1 + x + xF(x) - x + x^2 F(x)
\end{aligned}
$$

So $F(x)(1 - x - x^2) = 1 + x - x$
So
$$F(x) = \frac{1}{1 - x - x^2}$$
.

Geometric Series

$$G = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n$$

$$tG = t + t^2 + t^3 + \ldots$$

$$G - tG = 1$$

So
$$G = \frac{1}{1 - t}$$

If $\lambda \in \mathbb{C}$ and $t = \lambda x : \frac{1}{1 - \lambda x} = \sum_{n=0}^{\infty} \lambda^n x^n$
How to apply this to $F(x) = \frac{1}{1 - x - x^2}$?
Factor the denominator $1 - x - x^2 = (1 - \alpha)(1 - \beta x)$ for some $\alpha, \beta \in \mathbb{C}$.
(α, β are called inverse roots)
Now
$$F(x) = \frac{1}{(1 - \alpha x)(1 - \beta x)} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}$$
for some $A, B \in \mathbb{C}$. Why? Partial Fractions.

Determine α, β, A, B. Then

$$\begin{aligned} F(x) &= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\ &= A\sum_{n=0}^{\infty} \alpha^n x^n + B\sum_{n=0}^{\infty} \beta^n x^n \\ &= \sum_{n=0}^{\infty} \left(A\alpha^n + B\beta^n\right) x^n \end{aligned}$$

So $f_n = A\alpha^n + B\beta^n$ for all $n \geq 0$.

$$1 - x - x^2 = (1-\alpha x)(1-\beta x)$$

Subs $y = \frac{1}{x}$, multiply by y^2.

$$y^2 - y - 1 = (y-\alpha)(y-\beta)$$

$$\alpha, \beta = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = \frac{1}{1-x-x^2}$$

Clear the denominator.

$$A(1-\beta x) + B(1-\alpha x) = 1$$

$$(A+B) - (A\beta + B\alpha)x = 1$$

Compare coefficients of powers of x:

$$A + B = 1$$

$$A\beta + B\alpha = 0$$

Solve for A, B by linear algebra.

$$A\beta + B\beta = \beta$$

$$B\left(\beta - \alpha\right) = \beta$$

$$B = \frac{\beta}{\beta - \alpha}$$

$$A\alpha + B\alpha = \alpha$$

$$A\left(\alpha-\beta\right)=\alpha$$
$$A=\frac{\alpha}{\alpha-\beta}$$

See the notes

$$f_n=\frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n+\frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

2 January 8th

Natural Numbers

$$\mathbb{N}=\{0,1,2,3,\dots\}$$

include zero.

Factorials

For $n\in\mathbb{N}$:

$$n!=1\cdot 2\cdot 3\cdot\dots\cdot n$$

Binomial Coefficients

For $n,k\in\mathbb{N}$

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

Binomial Theorem

For $n\in\mathbb{N}$:

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$$

Binomial Series

For integers $t\geq 1$,

$$\frac{1}{(1-x)^t}=\sum_{n=0}^{\infty}\binom{n+t-1}{t-1}x^n$$

Example:

$$\frac{1}{(1+3x)^3}=\sum_{n=0}^{\infty}\binom{n+2}{2}(-3x)^n$$
$$\binom{n+2}{2}=\frac{(n+2)!}{2!n!}=\frac{(n+2)(n+1)}{2}$$

So

$$\frac{1}{(1+3x)^3}=\frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)3^n(-1)^nx^n$$

Combinatorial Proofs:

Let S,T be sets. Let $f:S\to T$ be a function.

- f is injective if for all $s, s' \in S$: if $f(s) = f(s')$, then $s = s'$.

  (Every element of T is the image of **at most** one element of S).

- f is surjective if for all elements $t \in T$, there exists $s \in S$ such that $f(s) = t$.

  (Every element of T is the image of at least one element of S.)

- f is bijective if it's injective and surjective.

$f : S \to T$ is a bijection, then for every $t \in T$, there is **exactly one** $s \in S$ such that $f(s) = t$.

Inverse bijection:

$$f^{-1} : T \to S$$

defined by $f^{-1}(t) = s$ if and only if $f(s) = t$.

Clearly $\left(f^{-1}\right)^{-1} = f$.

S and T are **equicardinal** if there is a bijection $f : S \to T$.

Notation: $S \rightleftharpoons T$.

Exercise:

$\rightleftharpoons$ is an equivalence relation.

A set S is infinite if S is equicardinal with a proper subset of S.

(i.e $T \subseteq S$ and $\emptyset \neq T \neq S$)

Example:

$\mathbb{N}$ is infinite, because

$$\mathbb{N} \rightleftharpoons \{0, 2, 4, 6, \dots\}$$

by $n \mapsto 2n$.

Otherwise, S is finite.

Cardinality of Finite Sets

$$|S| = \sum_{s \in S} 1$$

Unions of Sets

$$|S \cup T| = |S| + |T| - |S \cap T|$$

(For 3 or more sets: Inclusion — Exclusion).

Disjoint Unions

$$S \cap T = \emptyset.$$

$$|S \cup T| = |S| + |T|.$$

Cartesian Products

$$S \times T = \{(s, t) : s \in S \text{ and } t \in T\}$$

Exercise:

For finite sets, S, T

$$|S \times T| = |S| \cdot |T|$$

Lists:

A list of a set S is a sequence $a_1, a_2, \ldots, a_n$ in which each element of S occurs exactly once.

Note: in this case, $|S| = n$.

Examples:

$$S = \{1, A, \#\}$$

List of S:
$1, A, \#,$
$1, \#, A,$
$A, 1, \#,$
$\ldots$

Proposition:

If $|S| = n$, then S has $n!$ lists.
Let $\mathcal{L}(S)$ be the set of lists of S.
We prove $|\mathcal{L}(S)| = n!$ by induction on n.

Basis

$n = 0, 1, 2$, trivial.

Step:

Notice that

$$\mathcal{L}(S) \rightleftharpoons \bigcup_{s \in S} \{s\} \times \mathcal{L}(S \setminus \{s\})$$

$$a_1 a_2 \ldots a_n \mapsto (a_1, a_2 a_3 \ldots a_n)$$

Note that $\bigcup_{s \in S}$ is a disjoint union here.
By sums and products.

$$|\mathcal{L}(S)| = \sum_{s \in S} 1 \cdot |\mathcal{L}(S \setminus \{s\})| = \sum_{s \in S} (n-1)! = (n-1)! \sum_{s \in S} 1 = n!$$

by induction

3 January 10th

Partial Lists

Let S be a finite set. $|S| = n$.
Let $k \in \mathbb{N}$.

A partial list of S of length k is a sequence $a_1, a_2 \ldots, a_k$ of elements of S, each element of S occuring at most once.

Let $\mathcal{L}(S, k)$ be the set of partial lists of S of length k.
If $k > n$, then

$$\mathcal{L}(S, k) = \emptyset$$

Proposition:

For $0 \leq k \leq n$: $|\mathcal{L}(S, k)| = n(n-1) \cdots (n-k+1)$

Proof:

Fix $k \in \mathbb{N}$. Go by induction on $n = |S|$.

Basis:
$n = k$. $\mathcal{L}(S,n) = \mathcal{L}(n)$. and $|\mathcal{L}(S,n)| = n!$
Inductive Step:

$$\mathcal{L}(S,k) \rightleftharpoons \bigcup_{s \in S} (\{s\} \times \mathcal{L}(S \setminus \{s\}, k-1))$$

By Induction:

$$\begin{aligned} |\mathcal{L}(S,k)| &= \sum_{s \in S} |\mathcal{L}(S \setminus \{s\}, k-1)| \\ &= (n-1)(n-2)\dots((n-1)-(k-1)+1)) \sum_{s \in S} 1 \\ &= n(n-1)\cdots(n-k+1) \end{aligned}$$

k-element subsets
Let $\mathcal{B}(S,k)$ be the set of all k-element (unordered) subsets of S.
Lemma
If $S \rightleftharpoons T$, then

$$\mathcal{B}(S,k) \rightleftharpoons \mathcal{B}(T,k)$$

Proof:
Let $f : S \to T$ be a bijection.
Then $F : \mathcal{B}(S,k) \to \mathcal{B}(T,k)$ is a bijection.
Let $R \subseteq S$ be a k-element subset of S.
Define

$$F(R) = \{f(r) : r \in R\}$$

Apply this construction to f^{-1} to get F^{-1} (You check the details).
Corollary
There is a function $b(n,k)$ such that if $0 \le k \le n$ and $|S| = n$,
Then

$$|\mathcal{B}(S,k)| = b(n,k)$$

Proposition:
For $0 \le k \le n$, we have $b(n,k) = \binom{n}{k}$.
Proof:
Construct a partial list, $a_1 a_2 \dots a_k$ of S of length k as follows:

- Choose a k-element subset $R \subseteq S$
- Choose a list from the set $\mathcal{L}(R)$.

This produces every partial list in $\mathcal{L}(S,k)$ exactly once each.

$$\mathcal{L}(S,k) = \bigcup_{R \in \mathcal{B}(S,k)} \mathcal{L}(R)$$

Taking cardinalities.

$$|\mathcal{L}(S,k)| = \sum_{R\in\mathcal{B}(S,k)} |\mathcal{L}(R)|$$

$$n(n-1)\cdots(n-k+1) = \sum_{R\in\mathcal{B}(n,k)} k!$$

$$\frac{n!}{(n-k)!} = k! \cdot \sum_{R\in\mathcal{B}(S,k)} 1$$

So

$$b(n,k) = |\mathcal{B}(S,k)| = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Multisets

Informally, a "set with repeated elements". Fix a positive integer $t \geq 1$, the number of types of element.

For $1 \leq i \leq t$, let $m_i \in \mathbb{N}$ be the number of elements of the i-th type.

$$\mu = (m_1, m_2, \ldots, m_t) \in \mathbb{N}^t$$

is a multiset with t types, of size $|\mu| = m_1 + m_2 + \cdots + m_t$

Examples:

Skittles $t = 5$, types R, G, Y, O, P

$$\{R, O, R, Y, G, G, O, P, Y, R\}$$

$$(3, 2, 2, 2, 1)$$

How many multisets are there of size $n \in \mathbb{N}$ with $t \geq 1$ types of element?

Answer:

$$\binom{n+t-1}{t-1}$$

Let $\mathcal{M}(n,t)$ be the set of multisets of size n with elements of t types.

Note that $\binom{n+t-1}{t-1} = |\mathcal{B}(n+t-1, t-1)|$

Where $\mathcal{B}(n+t-1, t-1)$ is the set of all $(t-1)$-element subsets of $\{1, 2, \ldots, n+t-1\}$

Define a bijection $\mathcal{M}(n,t) \rightleftharpoons \mathcal{B}(n+t-1, t-1)$ to prove the result.

Return to the previous example:

$n = 10, t = 5, n+t-1 = 14, t-1 = 4.$

Bijection:

$$\mathcal{B}(n+t-1, t-1) \to \mathcal{M}(n,t)$$

Draw a row of circles of length $n+t-1$.

$$\circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\circ$$

Cross out $t-1$ of them to indicate a subset R of $\{1, 2, \ldots, n+t-1\}$.

Let m_i be the number of circles between the $(i-1)$st and i-th crossed out circles for each $2 \le i \le t-1$

Let m_i be the number of circles before the first X.

Let m_t be the number of circles after the last X.

Let $\mu = (m_1, m_1, \ldots, m_t)$.

Claim:

This construction $R \mapsto \mu$ defined a bijection

$$\mathcal{B}(n+t-1, t-1) \rightleftharpoons \mathcal{M}(n, t)$$

What is the inverse bijection?

Start with $\mu = (m_1, m_2, \ldots, m_t) \in \mathcal{M}(n+t-1)$.

For $1 \le i \le t-1$, let $s_i = m_1 + m_2 + \cdots + m_i + i$

Let $R = \{s_1, s_2, \ldots, s_{t-1}\}$

Claim:

This construction, $\mu \mapsto R$ is the inverse bijection.

Example:

$n = 10, t = 5, \mu = (2, 3, 0, 1, 4)$

So $(s_1, \ldots, s_4) =$

$s_1 = 2+1 = 3$, $s_2 = 2+3+2 = 7, s_3 = 2+3+0+3 = 8, s_4 = 2+3+0+1+4 = 10$

$$R = \{3, 7, 8, 10\}$$

Conversely, $R = \{3, 7, 8, 10\}$

Picture here.

4 January 13th

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = \frac{2 - 7x + 7x^2}{1 - 4x + 5x^2 - 2x^3}$$

Recurrence Relation (Theorem 4.5)

Partial Fractions (Theorem 4.9)

Recurrence Relations

$$(1 - 4x + 5x^2 - 2x^3) \sum_{n=0}^{\infty} c_n x^n = 2 - 7x + 7x^2$$

$$= \sum_{n=0}^{\infty} c_n x^n - 4 \sum_{n=0}^{\infty} c_n x^{n+1} + 5 \sum_{n=0}^{\infty} c_n x^{n+2} - 2 \sum_{n=0}^{\infty} c_n x^{n+3}$$

$$= \sum_{n=0}^{\infty} c_n x^n - 4 \sum_{i=1}^{\infty} c_{i-1} x^i + 5 \sum_{j=2}^{\infty} c_{j-2} x^j - 2 \sum_{k=3}^{\infty} c_{k-3} x^k$$

By convention, let $c_n = 0$ if $n < 0$. Then continue

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} c_n x^n - 4\sum_{i=0}^{\infty} c_{i-1}x^i + 5\sum_{j=0}^{\infty} c_{j-2}x^j - 2\sum_{k=0}^{\infty} c_{k-3}x^k \\
&= \sum_{n=0}^{\infty} \left(c_n - 4c_{n-1} + 5c_{n-2} - 2c_{n-3}\right) x^n
\end{aligned}
$$

Compare coefficients on LHS and RHS.
For $n \in \mathbb{N}$,

$$
c_n - 4c_{n-1} + 5c_{n-2} - 2c_{n-3} = \begin{cases} 2 & n = 0 \\ -7 & n = 1 \\ 7 & n = 2 \\ 0 & n \geq 3 \end{cases}
$$

in which $c_n = 0$ if $n < 0$.
When $n = 0$,

$$c_0 = 2.$$

When $n = 1$,

$$c_1 - 4c_0 = -7$$

$$c_1 = -7 + 4 \cdot 2 = 1$$

When $n = 2$,

$$c_2 - 4c_1 + 5c_0 = 7$$

$$c_2 = 7 + 4 \cdot 1 - 5 \cdot 2 = 1$$

Initial Conditions.
When $n \geq 3$,

$$c_n = 4c_{n-1} - 5c_{n-2} + 2c_{n-3}$$

Recurrence relation.

n	0	1	2	3	4	5	6
c_n	2	1	1	3	9	...	...

Partial Fractions:

$$\sum_{n=0}^{\infty} c_n x^n = \frac{P(x)}{Q(x)}$$

Applies only when $\deg(P) < \deg(Q)$.

Also, assume that the constant term of $Q(x)$ is $Q(0) = 1$.
Factor $Q(x)$ to find its "inverse roots".

$$Q(x) = (1 - \lambda_1 x)^{d_1}(1 - \lambda_2 x)^{d_2} \cdots (1 - \lambda_s x)^{d_s}$$

$\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_s$ pairwise distinct nonzero complex numbers, $d_1, d_2, \cdots, d_s$ positive integers: $d_1 + d_2 + \cdots + d_s = d = \deg(Q)$

Then, there are d complex numbers

$$C_1^{(1)}, C_2^{(1)}, \ldots, C_{d_1}^{(1)}$$

$$C_1^{(2)}, C_2^{(2)}, \ldots, C_{d_1}^{(2)}$$

$$\vdots$$

$$C_1^{(s)}, C_2^{(s)}, \ldots, C_{d_1}^{(s)}$$

which are uniquely determined such that

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{s} \sum_{j=1}^{d_i} \frac{C_j^{(i)}}{(1 - \lambda_i x)^j}$$

Useful together with Binomial Series Expansion.

$$\frac{1}{(1 - \alpha x)^p} = \sum_{n=0}^{\infty} \binom{n + p - 1}{p - 1} \alpha^n x^n$$

Example:

$$\frac{P(x)}{Q(x)} = \frac{2 - 7x + 7x^2}{1 - 4x + 5x^2 - 2x^3}$$

Factor the denominator.
$Q(1) = 0$ so $x - 1$ is a factor.

$$\begin{aligned} 1 - 4x + 5x^2 - 2x^3 =& (1 - x)(1 - 3x + 2x^2) \\ &= (1 - x)(1 - x)(1 - 2x) \\ &= (1 - x)^2(1 - 2x) \end{aligned}$$

Inverse roots:
1 with multiplicity 2.
2 with multiplicity 1.
By Partial Fractions

$$\frac{P(x)}{Q(x)} = \frac{A}{(1 - x)} + \frac{B}{(1 - x)^2} + \frac{C}{(1 - 2x)}$$

Solve for A, B, C.
Clear the denominator,

$$2 - 7x + 7x^2 = A(1-x)(1-2x) + B(1-2x) + C(1-x)^2$$

Evaluate:

- At $x = 1$:
  $$2 - 7 + 7 = A \cdot 0 + B(-1) + C \cdot 0$$
  So $B = -2$.
- At $x = \frac{1}{2}$:
  $$2 - \frac{7}{2} + \frac{7}{4} = A \cdot 0 + B \cdot 0 + C(1 - \frac{1}{2})^2$$
  $$C = 1$$
- At $x = 0$:
  $$2 - 0 + 0 = A + B + C$$
  $$A = 2 - B - C = 3$$

$$\frac{P(x)}{Q(x)} = \frac{3}{1-x} - \frac{2}{(1-x)^2} + \frac{1}{1-2x}$$
$$3\sum_{n=0}^{\infty} x^n - 2\sum_{n=0}^{\infty} \binom{n+2-1}{2-1} x^n + \sum_{n=0}^{\infty} 2^n x^n$$
$$\sum_{n=0}^{\infty} (3 - 2(n+1) + 2^n) x^n$$

So for all $n \in \mathbb{N}$:

$$c_n = 2^n - 2n + 1$$

n	0	1	2	3	4	5	6
c_n	2	1	1	3	9		53

5 January 15th

Subsets and Indicator Functions

Let $\mathcal{P}(n)$: set of all subsets of $\{1, 2, \ldots, n\}$
$\{0,1\}^n$: set of binary sequences $b_1 b_2 \ldots b_n$ of length n.
Bijection

$$\mathcal{P}(n) \rightleftharpoons \{0,1\}^n$$
$$S \leftrightarrow \beta$$

Given $S \subseteq \{1, 2, \ldots, n\}$
Define $\beta = b_1 b_2 \ldots b_n$ by

$$b_i = \begin{cases} 0 & i \notin S \\ 1 & i \in S \end{cases}$$

THis construction defines a function $S \mapsto \beta$ from $\mathcal{P}(n)$ to $\{0,1\}^n$

Given $\beta = b_1 b_2 \ldots b_n$, define $S \subseteq \{1, 2, \ldots, n\}$ by $S = \{i \in \{1, 2, \ldots, n\} : b_i = 1\}$.

This defines a function

$$\beta \mapsto S$$

from $\{0,1\}^n$ to $\mathcal{P}(n)$.

Claim:

These are mutually inverse bijection.

- $S \mapsto \beta$, then $\beta \mapsto T$. Prove that $T = S$.
- $\beta \mapsto S$, then $S \mapsto \alpha$. Prove that $\alpha = \beta$.

Proof: (Exercise).

$\mathcal{B}(n,k)$ set of all k-element subsets of $\{1, 2, \ldots, n\}$.

$$\mathcal{P}(n) = \bigcup_{k=0}^{n} \mathcal{B}(n,k)$$

is a disjoint union. Taking cardinalities

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}$$

Binomial Theorem

Copy this argument, keeping track of the sizes of the subsets $S \subseteq \{1, 2, \ldots, n\}$ in the exponent of x (an "indeterminate")

$$\mathcal{P}(n) \rightleftharpoons \{0,1\}^n$$
$$S \leftrightarrow \beta = b_1 b_2 \ldots b_n$$
$$|S| = b_1 + b_2 + \cdots + b_n$$

Because of the bijection:

$$\sum_{S \in \mathcal{P}(n)} x^{|S|} = \sum_{\beta \in \{0,1\}^n} x^{b_1 + b_2 + \cdots + b_n}$$

Left Hand Side:

$$\sum_{S \in \mathcal{P}(n)} x^{|S|} = \sum_{k=0}^{n} \sum_{S \in \mathcal{B}(n,k)} x^{|S|} = \sum_{k=0}^{n} x^k \sum_{S \in \mathcal{B}(n,k)} 1$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k$$

Right Hand Side:

$$\begin{aligned}
\sum_{\beta \in \{0,1\}^n} x^{b_1+b_2+\cdots+b_n} &= \sum_{b_1=0}^{1} \sum_{b_2=0}^{1} \cdots \sum_{b_n=0}^{1} x^{b_1+b_2+\cdots+b_n} \\
&= \sum_{b_1=0}^{1} x^{b_1} \sum_{b_2=0}^{1} x^{b_2} \cdots \sum_{b_n=0}^{1} x^{b_n} \\
&= (1+x)(1+x)\cdot \cdots \cdot (1+x) \\
&= (1+x)^n
\end{aligned}$$

So

$$(1+n)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Binomial Series

Let $t \geq 1$ be an integer, $n \in \mathbb{N}$.

Let $\mathcal{M}(n,t)$ be the set of multisets of size n with elements of t types.

$$\mu = (m_1, m_2, \ldots, m_t)$$

$$|\mu| = m_1 + m_2 + \cdots + m_t = n$$

Let $\mathcal{M}(t) = \bigcup_{n=0}^{\infty} \mathcal{M}(n,t)$

We know that

$$|\mathcal{M}(n,t)| = \binom{n+t-1}{t-1}$$

Keep track of the size of each multiset $\mu \in \mathcal{M}(t)$ in the exponent of x.

$$\sum_{\mu \in \mathcal{M}(t)} x^{|\mu|} = \sum_{n=0}^{\infty} \sum_{\mu \in \mathcal{M}(n,t)} x^{|\mu|}$$

$$\sum_{n=0}^{\infty} x^n \sum_{\mu \in \mathcal{M}(n,t)} 1 = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

Notice that $\mathcal{M}(t) = \mathbb{N} \times \mathbb{N} \times \ldots \mathbb{N} = \mathbb{N}^t$

So

$$\sum_{\mu \in \mathcal{M}(t)} x^{|\mu|} = \sum_{(m_1,\ldots,m_t) \in \mathbb{N}^t} x^{m_1+m_2+\cdots+m_t} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_t}^{\infty} x^{m_1+m_2+\cdots+m_t}$$

$$\sum_{m_1=0}^{\infty} x^{m_1} \cdot \sum_{m_2=0}^{\infty} x^{m_2} \cdots \sum_{m_t=0}^{\infty} x^{m_t} = \frac{1}{(1-x)^t}$$

(By Geometric Series)

In conclusion, for integer $t \geq 1$:

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

Sets and Weight Functions, Generating Series

Let $\mathcal{A}$ be a set (of combinatorial objects that we want to count)

A weight function is a function $\omega : \mathcal{A} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, the set

$$A_n = w^{-1}(n) = \{\alpha \in \mathcal{A} : \omega(\alpha) = n\}$$

is finite.

Note that

$$\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$$

is a disjoint union.

The generating series of $\mathcal{A}$ with respect to ω is

$$A(x) = \Phi_{\mathcal{A}}(x) = \sum_{\alpha \in \mathcal{A}} x^{\omega(\alpha)}$$

Example:

- $\mathcal{A} = \mathcal{P}(n)$.
- $\mathcal{A} = \mathcal{M}(t)$

Proposition:

Let $\mathcal{A}$ be a set with a weight function $w : \mathcal{A} \to \mathbb{N}$.

If

$$A(x) = \sum_{\alpha \in \mathcal{A}} x^{w(x)} = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$a_n = |\mathcal{A}_n|$$

is the number of objects in $\mathcal{A}$ of weight n.

Proof:

$$A(x) = \sum_{\alpha \in \mathcal{A}} x^{w(\alpha)} = \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{A}_n} x^{w(\alpha)}$$

$$= \sum_{n=0}^{\infty} x^n \sum_{\alpha \in \mathcal{A}_n} 1$$

$$= \sum_{n=0}^{\infty} |\mathcal{A}_n| x^n$$

Sum Lemma and Product Lemma
If $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $w : A \cup B \to \mathbb{N}$ is a weight function.
Then
$$\Phi_{\mathcal{A}\cup\mathcal{B}}(x) = \Phi_{\mathcal{A}}(x) + \Phi_{\mathcal{B}}(x)$$
If $w : \mathcal{A} \to \mathbb{N}$ and $v : \mathcal{B} \to \mathbb{N}$.
Define $f : \mathcal{A} \times \mathcal{B} \to \mathbb{N}$ by $f(\alpha, \beta) = w(\alpha) + v(\beta)$.
And
$$\Phi^{f}_{\mathcal{A}\times\mathcal{B}}(x) = \Phi^{w}_{\mathcal{A}}(x) \cdot \Phi^{v}_{\mathcal{B}}(x)$$

6 January 17th

Set $\mathcal{A}$, weight function $\omega : \mathcal{A} \to \mathbb{N}$ (for all $n \in \mathbb{N} : \mathcal{A} = \{\alpha \in \mathcal{A} : \omega(\alpha) = n\}$ is finite).

Generating series
$$A(x) = \Phi_{\mathcal{A}}(x) = \sum_{\alpha\mathcal{A}} x^{w(\alpha)} = \sum_{n=0}^{\infty} |\mathcal{A}_n| x^n$$

Infinite Sum Lemma
Let $\{\mathcal{A}_j : j \in J\}$ be a collection of sets.
Let $\mathcal{B} = \bigcup_{j\in J} A_j$. Assume that this is a disjoint union. If $i \neq j$ then $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$.
Let $\omega : \mathcal{B} \to \mathbb{N}$ be a weight function.
(This restricts to a weight function on each $\mathcal{A}_j$)
Then
$$\Phi_{\mathcal{B}}(x) = \sum_{\alpha\in\mathcal{B}} x^{\omega(\alpha)} = \sum_{j\in J} \sum_{\alpha\in\mathcal{A}_j} x^{\omega(x)} = \sum_{j\in J} \Phi_{\mathcal{A}_j}(x)$$
Need disjoint union in the third equal sign
Product Lemma:
Let $\mathcal{A}, \mathcal{B}$ be sets with weight functions $\omega : \mathcal{A} \to \mathbb{N}$ and $v : \mathcal{B} \to \mathbb{N}$.
Define $\theta : (A \times B) \to \mathbb{N}$ by
$$\theta(\alpha, \beta) = \omega(\alpha) + v(\beta)$$
Then
$$\Phi_{\mathcal{A}\times\mathcal{B}}(x) = \Phi_{\mathcal{A}}(x) \cdot \Phi_{\mathcal{B}}(x)$$
Proof: (Notes)
String Lemma:
Let $\mathcal{A}$ be a set with weight function, $\omega : \mathcal{A} \to \mathbb{N}$ such that there are no elements of $\mathcal{A}$ of weight 0.
Let $\mathcal{A}^k = \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$
$\omega_k = \mathcal{A}^k \to \mathbb{N}$ defined by
$$\omega_k(\alpha_1, \alpha_2, \ldots, \alpha_k) = \omega(\alpha_1) + \cdots + \omega(\alpha_k)$$

By the Product Lemma:

$$\Phi_{\mathcal{A}^k}(x) = (\Phi_{\mathcal{A}}(x))^k$$

Notation:

$$\mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k$$

a disjoint union.

Define $\omega^* (\alpha_1, \alpha_2, \ldots, \alpha_k) = \omega(\alpha_1) + \cdots + \omega(\alpha_k)$

Then

$$\Phi_{\mathcal{A}^*}(x) = \sum_{k=0}^{\infty} \Phi_{\mathcal{A}^k} \Phi_{\mathcal{A}^k}(x) = \sum_{k=0}^{\infty} (\Phi_{\mathcal{A}}(x))^k = \frac{1}{1 - \Phi_{\mathcal{A}}(x)}$$

How do we know that ω^* is a weight function?

$$\mathcal{A} = \{0, 1\}$$

$$\omega(i) = i$$

In $\mathcal{A}^* : (0, 0, \ldots, 0) \in \mathcal{A}^k$.

Infinitely many $\sigma \in \mathcal{A}^*$ of weight 0.

ω^* is not a weight function.

The answer is: We don't.

Lemma:

$$\omega^* : \mathcal{A}^* \to \mathbb{N}$$

is a weight function if and only if $\mathcal{A}_0 = \emptyset$: there are no elements in $\mathcal{A}$ of weight 0.

Proof:

(Notes / Exercise).

Example:

$\mathcal{A} = \{0, 1\}, \omega(i) = i$

$$\Phi_{\mathcal{A}}(x) = x^0 + x^1 = 1 + x$$

$$\frac{1}{1 - \Phi_{\mathcal{A}}(x)} = \frac{1}{1 - (1 - x)} = -\frac{1}{x} = -x^{-1}$$

2.3 Compositions

Definition:

A **composition** $\gamma = (c_1, c_2, \ldots, c_k)$ is a finite sequence of positive integers each c_i is a **part**.

The **length** is k, the number of parts.

The **size** is $|r| = c_1 + c_2 + \cdots + c_k$.

Examples:

Composition of size 4:

$(4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1)$

Let $\mathcal{C}_n$ be the set of compositions of size n.
Let
$$\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$$
For all $n \in \mathbb{N}$, what is $|\mathcal{C}_n|$?
What about compositions in $\mathcal{C}$ of a given length $k \in \mathbb{N}$?

- $k = 0 : \epsilon = ()$ empty composition
  length 0, size 0, generating series $1x^0 = 1$.
- $k = 1$: $\gamma = (c)$ for some $c \in \{1, 2, \ldots, \} = \mathbb{P}$
  Generating series:
  $$\sum_{c=1}^{\infty} x^c = x^1 + x^2 + x^3 + \cdots = \frac{x}{1-x}$$
- For general $k \in \mathbb{N}$:
  Composition of length k is the set
  $$\mathbb{P}^k = \mathbb{P} \times \mathbb{P} \times \ldots \mathbb{P}$$
  $$|\gamma| = c_1 + c_2 + \cdots + c_k$$
  Product Lemma applies.
  Generating Series
  $$\left(\frac{x}{1-x}\right)^k$$

All compositions
$$\mathcal{C} = \bigcup_{k=0}^{\infty} \mathbb{P}^k$$
and $\mathbb{P}$ has no elements of weight 0.
String lemma applies.
$$\Phi_{\mathcal{C}}(x) = \sum_{k=0} \Phi_{\mathbb{P}^k}(x) = \sum_{k=0}^{\infty} \left(\frac{x}{1-x}\right)^k$$
$$= \frac{1}{1 - \left(\frac{x}{1-x}\right)} = \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x}$$
$$= 1 + \sum_{j=0}^{\infty} 2^j x^{j+1} = 1 + \sum_{n=1}^{\infty} 2^{n-1} x^n$$
In conclusion, for any $n \in \mathbb{N}$,
$$|\mathcal{C}_n| = \begin{cases} 1 & n = 0 \\ 2^{n-1} & n \geq 1 \end{cases}$$

7 January 20th

Compositions

$r = (c_1, c_2, \dots, c_k)$ a sequence of positive integers.
Set of all compositions is $\mathcal{C} = \mathbb{P}^* = \bigcup_{k=0}^{\infty} \mathbb{P}^k$ where $\mathbb{P} = \{1, 2, 3, \dots\}$
Generating series is $\sum_{k=0}^{\infty} \left(\sum_{p=1}^{\infty} x^p\right)^k$ By the Sum and Product Lemma.

$$= \sum_{k=0}^{\infty} \left(\frac{x}{1-x}\right)^k = \frac{1}{1-\left(\frac{x}{1-x}\right)} = \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x}$$

Variations on this theme:

- What are the allowed values for this single part?
- What are the allowed lengths of the composition?

Then apply Sum and Product Lemmas.
Examples:
A: compositions in which all parts are ≥ 3 (any length is okay).

- Allowed values for one part: $P = \{3, 4, 5, \dots\}$
  Generating series for one part

$$\sum_{p=3}^{\infty} x^p = x^3 + x^4 + x^5 + \cdots = \frac{x^3}{1-x}$$

- For $k \geq 0$ parts: Generating series $\left(\frac{x^3}{1-x}\right)^k$ by Product Lemma.
- $k \in \mathbb{N}$ is arbitrary.

$$A(x) = \sum_{k=0}^{\infty} \left(\frac{x^3}{1-x}\right)^k = \frac{1-x}{1-x-x^3} = 1 + \frac{x^3}{1-x-x^3}$$

Examples:
$\mathcal{B}$: compositions in which each part is $\equiv 1 \pmod 3$

- allowed parts: $P = \{1, 4, 7, 10, \dots\}$ Generating Series: $x + x^4 + x^7 + x^{10} + \cdots = \frac{x}{1-x^3}$
- For $k \in \mathbb{N}$ parts: generating series is $\left(\frac{x}{1-x^3}\right)^k$ by Product Lemma.
- So, by the Sum Lemma

$$\mathcal{B}(x) = \sum_{k=0}^{\infty} \left(\frac{x}{1-x^3}\right)^k = \frac{1}{1-\left(\frac{x}{1-x^3}\right)} = \frac{1-x^3}{1-x-x^3} = 1 + \frac{x}{1-x-x^3}$$

Notation:

For a power series $G(x) = \sum_{n=0}^{\infty} g_n x^n$.

Let $[x^n]G(x) = g_n$ denote the coefficient of x^n.

Notice that $[x^n]x^d G(x) = \begin{cases} 0 & n < d \\ [x^{n-d}]G(x) & n \geq d \end{cases}$

IN the two examples $\mathcal{A}$ and $\mathcal{B}$, if $n \geq 3$, then

$$[x^n]A(x) = [x^n]\left(1 + \frac{x^3}{1-x-x^3}\right) = [x^n]x^3 \frac{1}{1-x-x^3}$$

$$= [x^{n-3}]\frac{1}{1-x-x^3}$$

$$= [x^{n-2}]x\frac{1}{1-x-x^3}$$

$$= [x^{n-2}]\left(1 + \frac{x}{1-x-x^3}\right)$$

$$= [x^{n-2}]B(x)$$

Let $\mathcal{A}_n, \mathcal{B}_n$ be the compositions of size n in $\mathcal{A}$ or $\mathcal{B}$, respectively.

From (*) if $n \geq 3$, then

$$|\mathcal{A}_n| = |\mathcal{B}_{n-2}|$$

Huh!

Can you explain this combinatorially by finding a bijection $\mathcal{A}_n \rightleftharpoons \mathcal{B}_{n-2}$?

$$A(x) = \frac{1}{1 - \left(\frac{x^3}{1-x}\right)} = \frac{1-x}{1-x-x^3} = \sum_{n=0}^{\infty} a_n x^n$$

By Linear Recurrence Relations

$$a_n - a_{n-1} - a_{n-3} = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

(Where $a_n = 0$ if $n < 0$).

$$a_0 = 1$$

$$a_1 - a_0 = -1, a_1 = 0$$

$$a_2 - a_1 = 0, a_2 = 0$$

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	0	0	1	1	1	2	3	4	6	9

$\mathcal{A}_9$	$\mathcal{B}_7$
(9)	(7)
(6, 3)	(4, 1, 1, 1)
(3, 6)	(1, 4, 1, 1)
(5, 4)	(1, 1, 4, 1)
(4, 5)	(1, 1, 1, 4)
(3, 3, 3)	(1, 1, 1, 1, 1, 1, 1)

Subsets with Restrictions

Examples:

For $n \in \mathbb{N}$, how many subsets of $\{1, 2, \ldots, n\}$ are there with no two consecutive numbers (a and $a+1$)? Call it r_n.

Eg:

$n = 4$:

$$\emptyset$$
$$\{1\}, \{2\}, \{3\}, \{4\}$$
$$\{1,3\}, \{1,4\}, \{2,4\}$$

$$r_4 = 8$$

.

n	0	1	2	3	4	5
r_n	1	2	3	5	8	

Turn this question about subsets into a question about compositions.

Let $S \subseteq \{1, 2, \ldots, n\}$ with no two consecutive elements.

$$1 \leq s_1 < s_2 < \cdots < s_k \leq n$$

For convenience, let $s_0 = 0$ and $s_{k+1} = n + 1$.

For $1 \leq i \leq k+1$, let $c_i = s_i - s_{i-1}$. and $\gamma = (c_1, c_2, \ldots, c_{k+1})$.

Example:

$n = 11$ and $S = \{3, 4, 7, 9\}$.

$$s_0 < s_1 < s_2 < s_3 < s_4 < s_5$$
$$0 < 3 < 4 < 7 < 9 < 12$$

$$\gamma = (3, 1, 3, 2, 3)$$

From the pair (n, S), we produced γ.

Claim: This is a bijection between the set $\mathcal{U} = \{(n, S) : n \in \mathbb{N} \text{ and } S \subseteq \{1, 2, \ldots, n\}\}$ and the set $\mathcal{C} \setminus \{\epsilon\}$ of nonempty compositions.

$$\mathcal{U} \Rightarrow \mathcal{C} \setminus \{\epsilon\}$$

$$(n, S) \iff (c_1, c_2, \ldots, c_l) = \gamma$$

$$|S| = l - 1$$

Note that:

$$|\gamma| = \sum_{i=1}^{k+1} c_i$$

$$= \sum_{i=1}^{k+1} (s_i - s_{i-1})$$

$$= s_{k+1} - s_0$$

$$= (n + 1) - 0 = n + 1$$

8 January 22nd

$\mathcal{U} = \{(n, S) : n \in \mathbb{N} \text{ and } S \subseteq \{1, 2, \ldots, n\}\}$

$$\mathcal{C} \setminus \{e\} = \bigcup_{l=1}^{\infty} \mathbb{P}^l$$

where $\mathbb{P} = \{1, 2, 3, \ldots\}$ is the set of nonempty compositions.

Bijection

$$\mathcal{U} \rightleftharpoons \mathcal{C} \setminus \{\epsilon\}$$

$$(n, S) \iff \gamma = (c_1, c_2, \ldots, c_l)$$

From $\mathcal{U}$ to $\mathcal{C} \setminus \{\epsilon\}$
Input:
$n \in \mathbb{N}$ and $S \subseteq \{1, 2, \ldots, n\}$;
Say $S = \{s_1, s_2, \ldots, s_k\}$ where $1 \leq s_1 < s_2 < \cdots < s_k \leq n$
Let $s_0 = 0$ and $s_{k+1} = n + 1$.
Define $c_i = s_i - s_{i-1}$ for all $1 \leq i \leq k + 1$.
Output:

$$\gamma = (c_1, c_2, \ldots, c_{k+1})$$

From $\mathcal{C} \setminus \{\epsilon\}$ to $\mathcal{U}$
Input:

$$\gamma = (c_1, c_2, \ldots, c_l)$$

with $l \geq 1$ for $1 \leq i \leq l-1$, define $s_i = c_1 + c_2 + \cdots + c_i$

Output:

$$S = \{s_1, s_2, \ldots, s_{l-1}\}$$

and

$$n = |\gamma| - 1.$$

In this bijection

$$\mathcal{U} \rightleftharpoons \mathcal{C} \setminus \{\epsilon\}$$

$(n, S) \iff \gamma$
$|S| = l(\gamma) - 1$
$n = |\gamma| - 1$

Check:

$$(n, S) \mapsto \gamma$$

and

$$\gamma \to (m, R)$$

Then, $m = n$ and $R = S$.

Check:

$$\gamma \mapsto (n, S)$$

and

$$(n, S) \to \rho$$

Then, $\rho = \gamma$.

Pattern: Given some subset of pairs in $\mathcal{U}$.
What is the corresponding subset of $\mathcal{C} \setminus \{\epsilon\}$?

Example:

(n, S) is such that S has no two consecutive elements $(a, a+1)$

$$(n, S) \iff \gamma = (c_1, c_2, \ldots, c_l)$$

$$(8, \{1, 3, 7\}) \iff (1, 2, 4, 2)$$

Such pairs (n, S) correspond to compositions γ

- That are not empty
- First and last parts might be $= 1$
- other parts are ≥ 2.

$$\sum_{(n,S)} x^n = \sum_{\gamma} x^{|\gamma|-1}$$

Generating series for these compositions with respect to size $|\gamma|$.

Case Analysis by length

- $l = 1$: $\gamma = (c_1)$ with $c_1 \in \mathbb{P}$

  Generating series. $\sum_{c_1=1}^{\infty} x^{c_1} = \frac{x}{1-x}$

- $l = 2$: $\gamma = (c_1, c_2)$ with $c_1, c_2 \in \mathbb{P}$

  Generating Series: $\left(\frac{x}{1-x}\right)^2$

- $l \geq 3$: $\gamma = (c_1, c_2, \dots, c_{l-1}, c_l)$ with $c_1, c_l \in \mathbb{P}$ and $c_i \in \{2, 3, 4, \dots\}$ for $2 \leq i \leq l-1$

  $c_i \in Q = \{2, 3, 4, \dots\}$ for $2 \leq i \leq l-1$

  That is, $\gamma \in \mathbb{P} \times Q \times Q \times \cdots \times Q \times \mathbb{P}$

  Generating Series:

  By the Product Lemma:

$$\left(\frac{x}{1-x}\right)\left(\frac{x^2}{1-x}\right)\cdots\left(\frac{x^2}{1-x}\right)\frac{x}{1-x}$$

  Also works for $l = 2$.

By the Sum Lemma, since $l \geq 1$:

$$\begin{aligned}
\sum_{\gamma} x^{|\gamma|} &= \frac{x}{1-x} + \sum_{l \geq 2}\left(\frac{x}{1-x}\right)^2\left(\frac{x}{1-x}\right)^{l-2} \\
&= \frac{x}{1-x} + \left(\frac{x}{1-x}\right)^2 \sum_{j=0}^{\infty}\left(\frac{x^2}{1-x}\right)^j \\
&= \frac{x}{1-x}\left[1 + \frac{x}{1-x} \cdot \frac{1}{1-\left(\frac{x^2}{1-x}\right)}\right] \\
&= \frac{x}{1-x}\left[1 + \frac{x}{1-x-x^2}\right] \\
&= \frac{x}{1-x}\left[\frac{1-x^2}{1-x-x^2}\right] \\
&= \frac{x(1-x^2)}{(1-x)(1-x-x^2)} \\
&= \frac{x(1+x)}{1-x-x^2}
\end{aligned}$$

So

$$\sum_{(n,S)} x^n = \sum_{\gamma} x^{|\gamma|-1} = \frac{1+x}{1-x-x^2} = \sum_{n=0}^{\infty} g_n x^n$$

$$g_n - g_{n-1} - g_{n-2} = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

$g_0 = 1, g_1 = 2, g_n = g_{n-1} + g_{n-2}$ for $n \geq 2$.

n	0	1	2	3	4	5	6	7
g_n	1	2	3	5	8	13	21	34

Chapter 3: Binary Strings
A **binary string** is a finite sequence of bits.

$$\sigma = b_1 b_2 \dots b_k$$

with each bit $b_i \in \{0, 1\}$.
The **length** is $l(\sigma) = k$.
Binary strings of length k are in $\{0, 1\}^k$.
Since $k \in \mathbb{N}$, an arbitrary binary string is in $\bigcup_{k=0}^{\infty} \{0, 1\}^k = \{0, 1\}^*$.
General problem:
For some subset $\mathcal{L} \subseteq \{0, 1\}^*$, determine the generating series.

$$L(x) = \Phi_{\mathcal{L}}(x) = \sum_{\sigma \in \mathcal{L}} x^{l(\sigma)} = \sum_{n=0}^{\infty} |\mathcal{L}_n| x^n$$

where $\mathcal{L}_n = \{\sigma \in \mathcal{L} : l(\sigma) = n\}$.
Example:
If $\mathcal{L} = \{0, 1\}^*$, then $\mathcal{L}_n = \{0, 1\}^n$. So $|\mathcal{L}_n| = 2^n$.
So $\Phi_{\{0,1\}^*}(x) = \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$
For any $\mathcal{L} \subseteq \{0, 1\}^*$, $|\mathcal{L}_n| \geq 2^n$, so $l : \mathcal{L} \to \mathbb{N}$ is always a weight function.

9 January 24th

Binary String
A string $\sigma = b_1 b_2 \dots b_n$ in $\{0, 1\}^*$ is also called a "**word**".
A set of $\mathcal{L} \subseteq \{0, 1\}^*$ is also called a "language".
A language is **rational** if it is produced by a regular expression. (reg. exp.)
Regular Expression is defined recursively.

- $\epsilon, 0, 1$ are regular expressions.
- If A is a regular expression then so is A^*
- If A, B are regular expressions, then so are $A \cup B$ and AB.

Regular expressions are just strings of symbols.
Example:

$$(0 \cup 11)^*$$

A regular expression A produces a subset $\mathcal{A} \subseteq \{0, 1\}^*$ as follows.
(Shorthand: $A \rhd \mathcal{A}$)

- $\epsilon \triangleright \{\epsilon\}, 0 \triangleright \{0\}, 1 \triangleright \{1\}$
- If $A \triangleright \mathcal{A}$ and $B \triangleright \mathcal{B}$, then $A \cup B \triangleright \mathcal{A} \cup \mathcal{B}, AB \triangleright \{\alpha\beta : \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$

  Concatenation product of $\mathcal{A}$ and $\mathcal{B}$.

$$\mathcal{A}^k = \mathcal{A}\mathcal{A}\dots\mathcal{A}$$

  concatenation power
- If $A \triangleright \mathcal{A}$, then $A^* \triangleright \mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k$
-

Example:
$\mathcal{A} = \{010, 110\}, \mathcal{B} = \{11, 0010\}, \mathcal{A}\mathcal{B} = \{010\cdot 11, 010\cdot 0010, 110\cdot 11, 110\cdot 0010\}$ is a bijection with $\mathcal{A} \times \mathcal{B}$.

Example:
$\mathcal{C} = \{01, 011\}$, $\mathcal{D} = \{110, 10\}$,
$01 \cdot 110 = 011 \cdot 10$ is produced twice in $\mathcal{C}\mathcal{D}$.
$\mathcal{C}\mathcal{D} = \{01110, 0110, 011110\}$ is not in bijection with $\mathcal{C} \times \mathcal{D}$.

Example:
$(0 \cup 1)^*$ produces $(\{0\} \cup \{1\})^* = \{0, 1\}^*$.
All binary strings exactly once each.
$(0 \cup 01 \cup 1)^*$ produces $\{0, 1, 01\}^* = \{0, 1\}^*$
All binary strings - some are produced many times.

THe same set of string $\mathcal{L} \subseteq \{0, 1\}^*$ can be produced by many different regular expressions.

A regular expression is unambiguous if every string it produces is produced exactly once.

Unambiguousnessity can be checked recursively.

- $\epsilon, 0, 1$ are unambiguous. Assume that A, B are unambiguous.

  $A \cup B$ is unambiguous if and only if $\mathcal{A} \cap \mathcal{B} = \emptyset$

  AB is unambiguous if and only if $\mathcal{A}\mathcal{B} \rightleftharpoons \mathcal{A} \times \mathcal{B}$.

  A^* is unambiguous if and only if $\mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k$

  - All $\mathcal{A}^k \rightleftharpoons \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$
  - Union is disjoint

Example:

- $(0 \cup 1)^*$ is unambiguous.
- $(0 \cup 1 \cup 01)^*$ is ambiguous.

Facts we don't need

1. If $A \subseteq \{0,1\}^*$ is a rational language.

   Then there is some regular expression producing $\mathcal{A}$ that is unambiguous.

2. If $\mathcal{A}, \mathcal{B}$ are rational languages, then so is

   $\mathcal{A} \setminus \mathcal{B} = \{\sigma : \sigma \text{ is in } \mathcal{A} \text{ but not in } \mathcal{B}\}$

Exercise:

Show that (2) implies (1) (Recursively).

A regular expression leads to a rational function, $A \rightsquigarrow A(x)$ recursively as follows.

- $\epsilon \rightsquigarrow 1$, $0 \rightsquigarrow x$, $1 \rightsquigarrow x$

  Assume $A \rightsquigarrow A(x)$ and $B \rightsquigarrow B(x)$

  Then

  - $A^* \rightsquigarrow \frac{1}{1-A(x)}$
  - $A \cup B \rightsquigarrow A(x) + B(x)$
  - $AB \rightsquigarrow A(x)B(x)$

Theorem:

Let A be a regular expression producing $A \subseteq \{0,1\}^*$ leading to $A(x)$.

If A is unambiguous, then

$$\Phi_{\mathcal{A}}(x) = A(x)$$

Proof: (Exercise)

Sum, Product, String Lemmas.

Example:

$(0 \cup 1)^*$ and $(0 \cup 1 \cup 01)^*$ both produce $\{0,1\}^*$.

$(0 \cup 1)^*$ leads to $\frac{1}{1-(x+x)} = \frac{1}{1-2x}$

Great!

$(0 \cup 1 \cup 01)^*$ leads to $\frac{1}{1-(x+x+x^2)} = \frac{1}{1-2x-x^2}$

Bad!

Example:

$(0 \cup 11)^*$ is unambiguous leads to $\frac{1}{1-(x+x^2)} = \frac{1}{1-x-x^2}$

which strings are produced?

0010111001111001100	NO
0011011111100001111	YES

10 January 27th

Unambiguous Expressions

- Block Decompositions

$$0011011110011011110000110111001100$$

A **block** of $\sigma = b_1 b_2 \dots b_n$ is a maximal (nonempty) subsequence of consecutive equal bits.

$$00|11|0|1111|00|11|0|1111|0000|11|0|111|00|11|00$$

Every binary string in $\{0,1\}^*$ can be decomposed uniquely into its sequence of blocks.

Produce a string block-by-block.

  - A block of 1s : $\{1, 11, 111, \dots\}$ produced by 1^*1 or 11^* or $\{1\}\{1\}^*$
  - A block of 0s: 0^*0.
  - A block of 0s followed by a block of 1s: $0 * 01 * 1$
  - Repeat this pattern arbitrarily often: $(0^*01^*1)^*$
  - Maybe you start with 1s: $(\epsilon \cup 1^*1) \equiv 1^*$
  - Maybe you end with 0s: 0^*.

In summary,

$$1^* \left(0^*01^*1\right)^* 0^*$$

is an unambiguous expression for all of $\{0,1\}^*$.

$(0 \cup 1)^*$

It leads to:

$$\frac{1}{1-x} \cdot \frac{1}{1 - \left(\frac{x}{1-x} \cdot \frac{x}{1-x}\right)} \cdot \frac{1}{1-x} = \frac{1}{1-2x}$$

Generating series for all binary strings.

Example:

$$\mathcal{L} \subseteq \{0,1\}^*$$

no blocks of 0s of length 1.

Blocks of 1s: 1^*1

Blocks of 0s: $00^*0, 0^*00, 000^*$

$$1^* \left(0^*001^*1\right)^* (\epsilon \cup 0^*00)$$

block decomposition, hence unambiguous.

Leads to

$$\frac{1}{1-x}\cdot\frac{1}{1-\left(\frac{x^2}{1-x}\cdot\frac{x}{1-x}\right)}\cdot\left(1+\frac{x^2}{1-x}\right)$$

$$=\frac{1-x+x^2}{(1-x)^2-x^3}=\frac{1-x+x^2}{1-2x+x^2-x^3}$$

Use recurrence relations to calculate $|\mathcal{L}_{10}|$

Prefix Decomposition

Given a binary string σ, chop it into pieces after each occurrence of the bit 1.

$$0001|1|001|001|1|1|0001|1|0000$$

This can be done uniquely.

What do the pieces look like?

$$(0^*1)^*0^*$$

leads to

$$\frac{1}{1-\left(\frac{x}{1-x}\right)}\cdot\frac{1}{1-x}=\frac{1}{1-2x}$$

.

Prefix Decomposition : A^*B.

Either σ is produced by B or it has a (non-empty) prefix produced by A.

(Do check that it's unambiguous)

Examples:

$\mathcal{L}\subseteq\{0,1\}^*$ no blocks of 0s of length one, again.

adapt either

$$(0^*1)^*\,0^* \text{ or } (1^*0)^*\,1^*$$

Let's try $(0^*1)^*\,0$.

$$1|1|001|0001|1|001|1|00$$

What do the pieces look like?

End piece:

$$\epsilon\cup 0^*00$$

Intial pieces:

$$[\epsilon\cup 000^*]1$$

$[(\epsilon\cup(0^*00)\,1)]^*\,(\epsilon\cup 0^*00)$ prefix decomposition for $\mathcal{L}$.

Leads to

$$\frac{1}{1-(1+\frac{x^2}{1-x})\cdot x}\cdot\left(1+\frac{x^2}{1-x}\right)$$

$$=\frac{1-x+x^2}{(1-x)-x(1-x+x^2)}=\frac{1-x+x^2}{1-2x+x^2-x^3}$$

Recursive Decomposition:

- More general than regular expressions.
- Can describe subsets of strings more general than rational languages.

Examples:

$$S = \epsilon \cup (0 \cup 1)\, S$$

defines S in terms of itself.

This produces every string in $\{0,1\}^*$ once each.

Leads to

$$S(x) = 1 + (x + x)\, S(x)$$
$$(1 - 2x) S(x) = 1$$
$$S(x) = \frac{1}{1 - 2x}$$

Examples:

$$\mathcal{A} = \{\epsilon, 01, 0011, 000111, 00001111, \dots\}$$

has generating series.

$$1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1 - x^2}$$

So does

$$\mathcal{B} = \{\epsilon, 01, 0101, 010101, \dots\}$$

$\mathcal{B}$ is a rational language produced by $(01)^*$.

But $\mathcal{A} = \bigcup_{k=0}^{\infty} 0^k 1^k$ is not rational.

But $A = \epsilon \cup 0A1$ describes $\mathcal{A}$ recursively.

11 January 29th

Examples:

Binary strings that don't contain 0110 as a substring. Call this set $\mathcal{A}$.

Modify a block decomposition:

$$0^*\, (1^* 1 0^* 0)\, 1^* 1$$

ϵ or a block of 0s.

$(1 \cup 1^* 111)$

A block of 1s that is not of length 2.

Block of 0s.

ϵ or a block of 1s.

11000111101 is not produced by $0^* \left((1 \cup 1^* 111)\, 0^* 0\right)^* 1^*$

How to fix this?

$1^* 0^* \left((1 \cup 1^* 111)^* \, 0^* 0\right)^* 1^*$ is ambiguous.

$(11 \cup \epsilon)\, 0^* \left((1 \cup 1^* 111)^* \, 0^* 0\right)^* 1$ is also ambiguous.

Modify the prefix.

- block of 0s (0^*0)
- 110^*0
- ϵ

$$(0^* \cup 110^*0)\,((1 \cup 1^*111)\,0^*0)^*\,1^*$$

is unambiguous.

This is a block decomposition for $\mathcal{A}$. So it is unambiguous. It leads to the generating series.

$$\begin{aligned} &\left(\frac{1}{1-x} + \frac{x^2 \cdot x}{1-x}\right) \frac{1}{1 - \left(1 + \frac{x^3}{1-x}\right)\left(\frac{x}{1-x}\right)} \cdot \frac{1}{1-x} \\ =&\frac{1+x^3}{(1-x)^2 - (x(1-x)+x^3)x} \\ =&\frac{1+x^3}{1-2x+x^2-x^2+x^3-x^4} \end{aligned}$$

$$A(x) = \frac{1+x^3}{1-2x+x^3-x^4}$$

Examples:

Try avoiding

$$00111000011010000$$

:)

Second method:

Recursion.

$\mathcal{A}$: no occurrence of 0110.

$\mathcal{B}$: exactly one occurrence of 0110 at the very end.

Notice that $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Unknown rational functions: $A(x), B(x)$.

Derive two equations in two unknowns, and solve.

First equation.

Consider a string $\sigma \in \mathcal{A} \cup \mathcal{B}$.

- maybe $\sigma = \epsilon$ is empty (Note: $\epsilon \in A$)
- If $\sigma \neq \epsilon$, then delete the last bit: $\sigma = \rho 1$ or $\sigma = \rho 0$ for some string $\rho \in \mathcal{A}$.

  So $A \cup B = \epsilon \cup A\,(0 \cup 1)$

  [Each string in $\mathcal{A} \cup \mathcal{B}$ is counted exactly once by this construction]

  So $A(x) + B(x) = 1 + 2xA(x)$.

Second equation:

Let $\sigma \in \mathcal{B} : \sigma = \alpha 0110$ for some $\alpha \in \mathcal{A}$.

So $\mathcal{B} \subseteq \mathcal{A}0110$.

What about the converse set inclusion: $\mathcal{A}0110 \subseteq \mathcal{B}$?

No! 011—0110 is in $\mathcal{A}0110$, but not in $\mathcal{B}$.

If $\alpha \in \mathcal{A}$ and $\alpha 0110$ is not in $\mathcal{B}$, then what does $\alpha 0110$ look like?

It has to contain a substring 0110 that is not at the very end.

Since 0110 does not occur in α, this "early" 0110 has to overlap the final 0110 non-trivially. (At least one bit but not all bits.)

Case analysis:

	σ
α	0 1 1 0
011	0 ...
01	01 ...
0	11 0 ...

The second overlap is possible.

For the rest, disagreements make these overlaps impossible.

In this case:

$$\sigma = \alpha 0110 = \beta 110$$

We saw that $\mathcal{B} \subseteq \mathcal{A}0110$. Conversely, $\mathcal{A}0110 \subseteq \mathcal{B} \cup \mathcal{B}110$

Let $\tau \in \mathcal{B}110$.

So $\tau = \alpha 0110|110$. Then **claim** $\alpha 011$ is in $\mathcal{A}$.

If not, then 0110 occurs in $\alpha 011$.

So $\mathcal{A}0110 = \mathcal{B}\,(\epsilon \cup 110)$

Second equation:

$$x^4 A(x) = B(x)(1 + x^3)$$

First equation:

$$A(x) + B(x) = 1 + 2xA(x)$$

$$B = \frac{x^4 A}{1 + x^3}$$

$$A + \frac{x^4 A}{1 + x^3} = 1 + 2xA$$

$$(1 + x^3)A + x^4 A = 1 + x^3 + 2xA(1 + x^3)$$

$$A(1 + x^3 + x^4 - 2x - 2x^4) = 1 + x^3$$

$$A(x) = \frac{1 + x^3}{1 - 2x + x^3 - x^4}$$

Finite State MAchines

Application 1: Excluded substrings

S a finite "alphabet" $S = \{0, 1\}$.

S^* all strings of letters from S.

$\mathcal{K}$ a finite subset of S^*

$A \subseteq S^*:$ all strings $\sigma \in S^*$ that do not contain any string in $\mathcal{K}$ as a substring.

$$|S| = d, |S^n| = d^n$$

$$\sum_{\sigma \in S^*} x^{l(\sigma)} = \frac{1}{1 - dx}$$

How to calculate $A(x) = \sum_{\alpha \in A} x^{l(\alpha)}$?

Example:

Strings in $\{a, b\}^*$ avoiding $abba$.

- Start with ϵ,
- build strings one letter at a time.
- Be careful if you are getting close to building a forbidden string.

Picture here.

Strings avoiding $abba$ correspond to ways of starting at ϵ and following the arrows in the transition diagram.

The number of steps = length of the string

(Can end anywhere)

Examples:

Strings in $\{a, b, c\}^*$ avoiding aa, cb, bcc, cab.

Transition table.

Transition Table

States	Next States
ϵ	a, b, c
a	aa, ab, ac
b	ba, bb, bc
c	ca, cb, cc
bc	bca, bcb, bcc
ca	caa, cab, cac

States: ϵ, single letters, and proper prefixes of forbidden strings.

Cross out the forbidden words, and we only need to keep track of the suffix of the words.

Pictures here.

Translation into algebra

Define a square matrix M indexed by states, $\sigma_1, \sigma_2, \ldots, \sigma_n$

$$M_{ij} = \begin{cases} 0 & \sigma_j \to \sigma_i \text{ is not allowed} \\ 1 & \sigma_j \to \sigma_i \text{ is allowed} \end{cases}$$

This is the transition matrix.

6×6 transition matrix.

$$M = \begin{array}{c|c|c|c|c|c} & \epsilon & a & b & ab & abb \\ \hline \epsilon & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 1 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 0 & 1 \\ ab & 0 & 1 & 0 & 0 & 1 \\ abb & 0 & 0 & 0 & 1 & 0 \end{array}$$

This is the transition matrix.

M_{ij} is the number of ways to get from state j to state i in exactly 1 step.

Lemma: For all $k \in \mathbb{N}$: $(M)_{ij}^{k}$ is the number of walks in the transition diagram from state j to state i with exactly k steps.

Proof:

Induct on k: $k = 0, M^0 = I$ $k = 1$, observation

Basis of induction.

Induction step:

$$\left(M^{k+1}\right)_{ij} = \sum_{h=1}^{n} (M_{ih}) \left(M^{k}\right)_{hj} = \sum_{h=1}^{n} (M_{ih})$$

Number of k-step walks from $\sigma_j \to \sigma_h$

$=$ the number of $k+1$-step walks $\sigma_j \to \sigma \to \sigma_i$.

$$\sum_{k=0}^{\infty} x^k M^k = (I - xM)^{-1} = A(x)$$

$A_{ij}(x)$ is the generating series for all walks in the transition diagram from state j to state i. (Keeping track of the length) in the exponent of x.

Forbidden *abba* example:

Starting state ϵ:

$$\underline{v}_{init} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Ending state arbitrary:

$$\underline{v}_{final} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Answer:

Generating series for strings in $\{a, b\}^*$ avoiding abba is

$$G(x) = \underline{v}_{final}^{T} \left(I - xM\right)^{-1} \underline{v}_{init}$$

12 Feburary 3rd

Application 2: Domino Tilings

Consider a $k \times n$ chessboard. Cover the squares with nonoverlapping dominos (2 by 1 rectangles)

In how many ways can this be done?

Case $k = 3$

See pictures.

States: A, B, B', C, C'

See pictures.

B and B' are related by symmetry. Also C and C'.

Three states

See pictures.

Transition matrix

$$T = \begin{bmatrix} t^3 & t & 0 \\ 2t^2 & 0 & t \\ 0 & t^2 & 0 \end{bmatrix}$$

T takes the place of xM from Friday's class.

$\left(T^k\right)_{ij}$ sums over all ways to go from state j to state i in k steps, keeping track of t^α when d dominoes have been used.

$$\sum_{k=0}^{\infty} T^k = (I - T)^{-1}$$

$\left(A^{-1}\right)_{ij}$ is the usm over all ways to go from state j to state i. (Keeping track of t^d when using d dominoes).

Answer:

$(I - T)^{-1}_{AA}$ is the generating series we want.

$(I - T)^{-1} = \frac{1}{\det(I-T)} \cdot \operatorname{adj}(I - T)$

$$I - T = \begin{pmatrix} 1 - t^3 & -t & 0 \\ -2t^2 & 1 & -t \\ 0 & -t^2 & 1 \end{pmatrix}$$

$$\begin{aligned} \det(I - T) &= t \begin{vmatrix} 1 - t^3 & -t \\ 0 & -t^2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 - t^3 & -t \\ -2t^2 & 1 \end{vmatrix} \\ &= -t^3 \left(1 - t^3\right) + \left(1 - t^3\right) \cdot 1 - 2t^3 \\ &= 1 - 4t^3 + t^6 \end{aligned}$$

$$\operatorname{adj}(I - T)_{AA} = \begin{vmatrix} 1 & -t \\ -t^2 & 1 \end{vmatrix} = 1 - t^3$$

$$D_3(t) = (I-T)^{-1}_{AA} = \frac{1-t^3}{1-4t^3+t^6}$$

$= \sum_{d=0}^{\infty} c_d t^d$ where c_d is the number of $3 \times n$ domino tilings with d dominos.
Note: $2d = 3n, n = \frac{2}{3}d$.
Let $t = x^{\frac{2}{3}}$.

$$D_3(x^{2/3}) = \frac{1-x^2}{1-4x^2+x^4}$$
$$= \sum_{n=0}^{\infty} g_n x^n$$

g_n = the number of domino tilings of a $3 \times n$ rectangle.

n	0	1	2	3	4	5	...
g_n	1	3	11	41	153	571	...

Picture here.
r_n: Irreducible pieces with n columns.

n	0	1	2	3	4	5	6	7	8
r_n	0	X	3	X	2	X	2	X	2

$$R(x) = 3x^2 + \frac{2x^4}{1-x^2} = \sum_{n=0}^{\infty} r_n x^n$$

$$\frac{1}{1-R(x)} = \frac{1-x^2}{(1-x^2)-(3x^2+3x^4-2x^4)} = \frac{1-x^2}{1-4x^2+x^4}$$

13 Feburary 5th

Application 3: Tessellation

Fix integers $d, k \geq 3$. Dissect the plane into k-gons, (polygon with k sides) so that every "vertex" (corner) is on exactly d of the polygons.

Example:
$d = 4, k = 4$
Square grid
Pictures here.
Let v_n be the number of vertexs that are n steps away from the base vertex. v_*.

$$\sum_{n=0}^{\infty} v_n x^n = 1 + 4x + 8x^2 + \dots$$

$$= 1 + 4\sum_{n=1}^{\infty} n \cdot x^n = 1 + \frac{4x}{(1-x)^2}$$

Examples:
$d = 6, k = 3.$
Triangular grid
Pictures here.

$$\sum_{n=0}^{\infty} v_n x^n = 1 + 6x + 12x^2 + \dots$$

$$= 1 + 6 \cdot \sum_{n=1}^{\infty} n \cdot x^n$$

$$= 1 + \frac{6x}{(1-x)^2}$$

Examples:
$d = 3, k = 6$
Hexagonal grid
Picture here.
$d = 3, k = 4$

$$1 + 3x + 3x^2 + x^3$$

$$d \geq 3$$

$\mathcal{H}_{d,k}$	3	4	5	6	7
3	tetrahedron	octagedron	Icosahedron	triangular grid	
4	cube	square grid			
5	Dodecahedron				
6	hexagonal grid				

Five Platonic Solids
Three Flat (”Euclidean”) grids
Rest is Hyperbolic Tessellations
Pictures here.
Examples:
$k = 4, d = 5$
Pictures.
v_n = the number of vertices that is n step away from base vertex v_*.

$$\sum_{n=0}^{\infty} v_n x^n = 1 + 5x +$$

Distance from v_* to vertex v is n. v has

- type A : if it has one neighbour at distance $n-1$.
- Type B : if it has 2 neighbouts at distance $n-1$.

v_* special type 0. "Origin".
For $n \geq 1$: a_n vertices of type A at distance n.
b_n vertices of type B at distance n.
Claim:
Every vertex other than v^* has type A or B.
Then $v_n = a_n + b_n$ for $n \geq 1$.
Recurrences.
For $n \geq 1$:

$$a_{n+1} =$$

$$b_{n+1} =$$

Population vector at distance n.
Three types $(0, A, B)$

$$p_n = \begin{bmatrix} 0_n \\ a_n \\ b_n \end{bmatrix}$$

$$p_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$p_1 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0 \\ 10 \\ 5 \end{bmatrix}$$

And so on.
The idea is to find this generating series.

$$\sum_{n=0}^{\infty} \begin{bmatrix} 0_n \\ a_n \\ b_n \end{bmatrix} x^n$$

14 Feburary 7th

Tessellation: See pictures.

At distance n

Origin O: $a_n = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}$

...

Succession rules:

$$\begin{array}{c|} \text{distance} \\ \vdots \\ n+2 \\ n+1 \\ n \\ n-1 \\ n-2 \end{array}$$

See pictures:

$$O \to 5A$$

$$A \to 2A + 2B$$

$$B \to 1A + 2B$$

But vertices of type B have 2 predecessors.

So this counts them twice each unless we include the factors of $\frac{1}{2}$.

So, for $n \geq 0$:

$$O_{n+1} = 0$$

$$a_{n+1} = 5O_n + 2a_n + b_n$$

$$b_{n+1} = a_n + b_n$$

Population vectors

$$p_n = \begin{bmatrix} O_n \\ a_n \\ b_n \end{bmatrix}$$

$$P_{n+1} = \begin{bmatrix} O_{n+1} \\ a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} O_n \\ a_n \\ b_n \end{bmatrix}$$

with $p_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

By induction on $n \in \mathbb{N}$, p_n is the population at distance n from the origin.

Total population at distance n is

$$v_n = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 5 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Generating series:

$$\sum_{n=0}^{\infty} v_n x^n$$

$$= \begin{bmatrix}1\\1\\1\end{bmatrix}^T \left(\sum_{n=0}^{\infty} x^n M^n\right) \begin{bmatrix}1\\0\\0\end{bmatrix}$$
$$= \begin{bmatrix}1\\1\\1\end{bmatrix}^T (I - xM)^{-1} \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$(I - xM) = \begin{bmatrix}1 & 0 & 0\\ -5x & 1-2x & -x\\ 0 & -x & 1-x\end{bmatrix}$$

$$\begin{aligned}\det(I - xM) &= \begin{vmatrix}1-2x & -x\\ -x & 1-x\end{vmatrix}\\ &= (1-2x)(1-x) - (-x)^2\\ &= 1 - 3x + 2x^2 - x^2\\ &= 1 - 3x + x^2 = D\end{aligned}$$

Let $A = (I - xM)^{-1}$.

Notice that $(I - xM)^{-1} \begin{bmatrix}1\\0\\0\end{bmatrix}$ is the first column of A.

$$A_{11} = \frac{1}{D}\begin{vmatrix}1-2x & -x\\ -x & 1-x\end{vmatrix} = 1$$

$$A_{21} = -\frac{1}{D}\begin{vmatrix}-5x & -x\\ 0 & 1-x\end{vmatrix} = -\left(\frac{-5x\,(1-x) - 0}{1-3x+x^2}\right) = \frac{5x - 5x^2}{1-3x+x^2}$$

$$A_{31} = \frac{1}{D}\begin{vmatrix}-5x & 1-2x\\ 0 & -x\end{vmatrix} = \frac{5x^2}{1-3x+x^2}$$

So

$$(I - xM)^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix}$$
$$= \frac{1}{1-3x+x^2}\begin{bmatrix}1-3x+x^2\\ 5x-5x^2\\ 5x^2\end{bmatrix}$$

So

$$V(x) = \sum_{n=0}^{\infty} v_n x^n = \frac{\left(1 - 3x + x^2\right) + \left(5x - 5x^2\right) + 5x^2}{1 - 3x + x^2}$$
$$= \frac{1 + 2x + x^2}{1 - 3x + x^2}$$

$$v_n - 3v_{n-1} + v_{n-2} = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 1 & n = 2 \\ 0 & n \geq 3 \end{cases}$$

$$v_0 = 1$$
$$v_1 - 3v_0 = 2 \to v_1 = 5$$
$$v_2 - 3v_1 + v_0 = 1 \to v_2 = 15 - 1 + 1$$
$$v_n = 3v_{n-1} - v_{n-2} (n \geq 3)$$

n	0	1	2	3	4	...
v_n	1	5	15	40	105	...

Extract formula via Partial Fractions:

$$\frac{1 + 2x + x^2}{1 - 3x + x^2} = 1 + \frac{5x}{1 - 3x + x^2}$$

...
Examples:
$d = 4, k = 5$
See pictures.

15 Feburary 10th

II. Graph Theory

Definition:
A **graph** is a pair of sets $G = (V, E)$

- An element of V is a **vertex** (plural: vertices)
- Elements of E are 2-elements subsets of V, called **edges**.

Examples:
$G = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 4\}, \{3, 5\}\})$
Picture of G:

We represent vertices by dots and edges by lines connecting the dots.
See Pictures.
Handshake Lemma
For $v, w \in V$, we also write vw for the edge $\{v, w\}$.
The degree of v is the number of edges that contain v denoted $\deg(v)$.
$v, w \in V$ are **adjacent**, or neighbours if $vw \in E$.
$v \in V$ and $e \in E$ are **incident** when $v \in e$, v is an end of e.
Degree Sequence of G is the multiset of vertex degrees (usually given as a sorted list)
See Pictures.
Same degree sequence doesn't need to look the same.
Same degree sequence but the "pattern of connections" are different.
Theorem: (Handshake Lemma)
Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|$$

Proof:
Consider the set

$$X = \{(v, e) \in V \times E : v \text{ is incident with } e\}$$

Count $|X|$ in two ways

$$\begin{aligned} |X| &= \sum_{v \in V} |\{(w, f) \in X : w = v\}| \\ &= \sum_{v \in V} \deg(v) \end{aligned}$$

$$\begin{aligned} |X| &= \sum_{e \in E} |\{(w, f) \in X : f = e\}| \\ &= \sum_{e \in E} 2 \\ &= 2 \cdot |E|. \end{aligned}$$

QED.
Corollary:
In a graph G, the number of vertices of odd degree is even.
(Handshake lemma modulo 2)
Examples:

- Empty graph $(\emptyset, \emptyset)$

- Edgeless graphs $(V, \emptyset)$
- Complete graphs $K_V = (V, \{vw : v, w \in V \text{ and } v \neq w\})$

  $K_n = K_{\{1,2,\ldots,n\}}$

$K_0 = (\emptyset, \emptyset)$.
Picture here.
Paths:
P_n for $n \geq 1$.

$$V(P_n) = \{1, 2, \ldots, n\}$$
$$E(P_n) = \{\{i, i+1\} : 1 \leq i \leq n-1\}$$

Picture here.
Cycles:
C_n for $n \geq 3$

$$V(C_n) = \{1, 2, \ldots, n\}$$
$$E(C_n) = E(P_n) \cup \{\{1, n\}\}$$

Picture here.
Definition:
Let $G = (V, E)$ and $H = (W, F)$ be graphs.
An **isomorphism** from G to H is

- a bijection $f : V(G) \to V(H)$ such that
- $\forall_{v,w} \in V(G)$: $\{f(v), f(w)\} \in E(H)$ if and only if $\{v, w\} \in E(G)$.

If there is an isomorphism from G to H, then G is isomorphic to H, denoted $G \cong H$.
See Picture here.

16 Feburary 12th

Let G and H be graphs. Assume that $f : V(G) \to V(H)$ is an isomorphism.
Necessary conditions on f

- If $v \in V(G)$ and $w = f(v)$, then $\deg_H(w) = \deg_G(v)$.

  Because f restricts to a bijection from the neighbours of v in G to the neighbours of w in H.

  Set of neighbours $N_G(v) = \{u \in V(G) : uv \in E(G)\}$
- If $G \cong H$, then they have the same degree sequence.

  Terminology:

  Given a graph $G = (V, E)$ and subset $W \subseteq V$ of vertices, the subgraph of G induced by W has vertex-set W and

edge-set $\{e \in E(G) : e \subseteq W\}$

Denoted by $G[W]$ or $G|_W$.

- If $f : G \to H$ is an isomorphism, then for all natural numbers $d \in \mathbb{N}$, f restricts to an isomorphism from the subgraph of G induced by the vertices of degree d to the corresponding subgraph of H.

  See pictures.

Structures inside graphs
Let $G = (V, E)$ be a graph.
A subgraph of G is a pair $H = (W, F)$ such that

- $W \subseteq V$
- $F \subseteq E$
- (W, F) is a graph. (That is, if $e \in F$ then $e \subseteq W$).

Pictures here.
$(\emptyset, \emptyset)$ is always a subgraph. (V, E) is always a subgraph.
All others are proper subgraphs.
$G[W]$ for $W \subseteq V$ is an **induced** subgraph.
$H = (W, F)$ is a **spanning** subgraph if $W = V$. (That is, H uses all vertices of G)
Edge-Deletion
For $S \subseteq E$, let $G \setminus S = (V, E \setminus S)$.
If $S = \{e\}$ write $G \setminus e$ instead of $G \setminus \{e\}$.
Vertex-Deletion
For $S \subseteq V$, let $G \setminus S = G[V \setminus S]$
If $S = \{v\}$, write $G \setminus V$ instead of $G \setminus \{v\}$.
A spanning cycle is called a **Hamilton** cycle.
A **grid** is a "product" of two paths: $P_r \square P_s$
Pictures here.
$V(G \square H) = V(G) \times V(H)$
$E(G \square H) = ...$
Which grids have Hamilton cycles?
Pictures.

17 Feburary 14th

Conjecture
$P_r \square P_s$ is Hamiltonian if and only if rs is even.

$$V\left(P_r \square P_s\right) = \{1, 2, \ldots, r\} \times \{1, 2, \ldots, s\}$$

$$\{(x,y),(a,b) \in E\}$$

iff

$$(x-a)^2 + (y-b)^2 = 1$$

Assume that r is even.

If rs is even, then assume that r is even (by symmetry). Describe a Hamilton cycle in $P_r \square P_s$ constructively.

If rs is odd, then we have to show that there is no Hamilton cycle in $P_r \square P_s$.

Bipartite Graphs

Let $G = (V, E)$ be a graph.

A **bipartition** of G is a pair (A, B) of subsets $A \subseteq V, B \subseteq V$ such that

- $A \cup B = V$ and $A \cap B = \emptyset$
- every edge $e \in E$ has one end in A and one end in B. ($e \cap A \neq \emptyset, e \cap B \neq \emptyset$)

A graph that has a bipartition is a bipartite graph.

Example:

$P_r \square P_s$ is bipartite.

Let $A = \{(x,y) \in V : x+y \text{ is even}\}$ $B = \{(x,y) \in V : x+y \text{ is odd}\}$

Check: this is a bipartition of $P_r \square P_s$.

Bipartite Handshake Lemma

Let $G = (V, E)$ be a graph with bipartition (A, B). Then

$$\sum_{v \in A} \deg(v) = |E| = \sum_{w \in B} \deg(w)$$

Corollary:

Let G be bipartite and regular of degree $d \geq 1$.

G is **regular** if all vertices have the same degree.

Then $|V(G)|$ is even.

Proof:

$$d|A| = \sum_{v \in A} \deg(v) = \sum_{w \in B} \deg(w) = d|B|$$

Since $d \geq 1$, we get $|A| = |B|$.

So $|V| = |A| + |B| = 2 \cdot |A|$.

Lemma:

Let G be bipartite. Then every subgraph of G is bipartite.

Proof:

Let (A, B) be a bipartition of G. Let $H = (W, F)$ be a subgraph of G.

Now, $(A \cap W, B \cap W)$ is a bipartition of H.

Corollary:

If G is bipartite and Hamiltonian, then $|V(G)|$ is even.

Proof:

Let C be a Hamiltonian cycle of G.

Then $V(C) = V(G)$ because C is a spanning subgraph of G. Since G is bipartite, C is bipartite.

Since C is a cycle, C is 2-regular.

By Corollary 1, $|V(C)|$ is even.

Finally, if rs is odd, then $P_r \Box P_s$ is not Hamiltonian.

Corollary 3: C_n is bipartite if and only if n is even.

- (C_n is a Hamilton cycle of itself, so if it is bipartite then n is even)
- Conversely, $V(C_n) = \{1, 2, \ldots, n\}, E(C_n) = \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$

Picture here.

If n is even, then $A = \{1, 3, 5, ..., n-1\}, B = \{2, 4, 6, \ldots, n\}$ is a bipartition, (A, B) of C_n.

Corollary:

If G contains an odd cycle, then G is not bipartite.

The converse is also true.

(Proof in a couple of weeks)

Walks, Paths and Connectedness

Let $G = (V, E)$ be a graph.

A **walk** in G is a sequence of vertices $W = (v_0 v_1 v_2 \ldots v_k)$ in which $v_{i-1} v_i \in E$ for all $1 \leq i \leq k$.

Picture here.

(qyrywxcxqrcxwp)

Path, walk, trails.

18 Feburary 24th

Walks, Paths and Cycles

$G = (V, E)$ a graph.

A **walk** is a sequence of vertices $W = (v_0 v_1 v_2 \ldots v_k)$ such that $v_{i-1} v_i \in E$ for all $1 \leq i \leq k$.

Each $v_{i-1} v_i$ is a step of W.

Length of W is $l(W) = k$, number of steps.

A **path** is a walk with no repeated vertices. (if $0 \leq i < j \leq k$, then $v_i \neq v_j$)

A **cycle** is a walk with no repeated vertices except that $v_0 = v_k$, and $k \geq 3$. (if $0 \leq i < j \leq k$ and $v_i = v_j$ then $i = 0$ and $j = k$)

A walk W is **supported** on the subgraph with

- Vertices $\{v_0, v_1, v_2, \ldots, v_k\}$
- Edges $\{v_0 v_1, v_1 v_2, \ldots, v_{k-1} v_k\}$

Also,

- Paths are supported on paths.
- Cycles are supported on cycles.

$W = (yzszs)$ is supported on a path, but not a path.
$W = (crsdcr)$ is supported on a cycle, but not a cycle.
Theorem: ”Shortest walks are paths”
Let $G = (V, E)$ be a graph.
Let $v, w \in V$. Let W be a (v, w)-walk of minimum length.
Remark: A (v, w)-walk is from $v_0 = v$ to $v_k = w$.
Then W is a path.
Proof:
Let $W = (v_0 v_1 v_2 \dots v_k)$ be a (v, w)-walk of minimum length.
Suppose W is not a path. There exist $0 \leq i < j \leq k$ with $v_i = v_j$.
Now $Z = (v_0 v_1 \dots v_i = v_j v_{j+1} \dots v_k)$ is a walk of length $l(Z) = l(W) - (j - 1) < l(W)$ from $v_0 = v$ to $v_k = w$.
This contradiction shows that W is a path.
Proposition: (”Two paths make a cycle”)
Let $G = (V, E)$ be a graph.
Let $v, w \in V$ be vertices. Let W, Z be distinct paths from v to w.
Then there is a cycle contained in the union of the supports of W and Z.
Proof:
Let $W = (v_0 v_1 v_2 \dots v_k)$ and $Z = (z_0 z_1 z_2 \dots z_l)$ be distinct (v, w)-paths. $(W \neq Z)$.
Since $W \neq Z$, there is an index $0 \leq a < \min\{k, l\}$ such that $v = v_0 = z_0, v_1 = z_1, \dots, v_a = z_a$ but $v_{a+1} \neq z_{a+1}$.
Let $a < b \leq k$ be the smallest index after a such that $v_b = v_c$ is also on Z.
b exists since $v_k = z_l$. Note: $a + 1 \leq c \leq l$ since W is a path.
Claim: $(v_a v_{a+1} \dots v_b z_{c-1} z_{c-2} \dots z_a)$ is a cycle.
Ckeck: No repeated vertices except $v_a = z_a$.

19 Feburary 26th

Proposition

Let $G = (V, E)$ be a nonempty graph. If $\deg(v) \geq 2$ for all $v \in V$, then G contains a cycle.
Proof:
Let P be a path in G that is as long as possible. (G contains a path since G is not empty)

$$P = (v_0 v_1 v_2 \dots v_k)$$

since all vertices have degree ≥ 2, the length of P is at least 2.
Since $\deg(v_k) \geq 2$, v_k has a neighbour $w \neq v_{k-1}$.
If w is not on P, then

$$(v_0 v_1 \dots v_k w)$$

is a path that is longer than P. Contradiction!

So $w = v_i$ for some $0 \leq i \leq k-2$.

Now, $C = (v_i v_{i+1} \dots v_{k-1} v_k v_i)$ is a cycle.

Connectedness

Let $G = (V, E)$ be a graph.

Let $v, w \in V$.

Say that v reaches w when there exists a (v, w)-walk in G.

(write vRw for short)

This is an equivalence relation on V.

- Reflexive: vRv
- Symmetric: If vRw, then wRv.
- Transitive: If vRw, wRz, then vRz.

Let $U_1, U_2, \dots, U_c$ be the equivalence classes of R on V.

Each $U_i \neq \emptyset, U_i \cap U_j = \emptyset$ if $i \neq j$, $U_1 \cup U_2 \cup \dots U_c = V$

The (connected) components of G are the subgraphs.

$$G_i = G[U_i]$$

induced by the subsets U_i.

Example:

$C_{12}(2, 4)$

Each connected component is not empty.

G is **connected** if G has exactly one connected components.

So $(\emptyset, \emptyset)$ is not connected.

Proposition:

Let $G = (V, E)$ be a graph.

G is connected if and only if there is a vertex of $v \in V$ such that for all $w \in V$, there is a (v, w)-path.

Proof: (Exercise.)

Connectedness and Cuts

Let $G = (V, E)$ be a graph.

For $S \subseteq V$, let the boundary of S to be the set $\partial S = \{e \in E : |e \cap S| = 1\}$ of edges with exactly one end in S.

(Also called the **cut** of S)

Example:

Picture here.

Theorem:

Let $G = (V, E)$ be a nonempty graph, then G is connected if and only if for every $\emptyset \neq S \subsetneq V$, the boundary $\partial S \neq \emptyset$ is not empty.

Proof:

First, assume that G is connected and $\emptyset \neq S \subsetneq V$.

Let $v \in S$ and $w \notin S$. Since G is connected.

There is a (v, w)-walk, $W = (v_0 v_1 \dots v_k)$.

Since $v_0 = v \in S$ and $v_k = w \notin S$.
There is an index $1 \leq i \leq k$ such that $v_{i-1} \in S$ and $v_i \notin S$.
Now, $v_{i-1}v_i \in \partial S$.
Second, assume that G is not connected.
Since $G \neq (\emptyset, \emptyset)$, it has at least two connected components.
Let S be the set of vertices of one component of G. Then $\emptyset \neq S \subsetneq V$ and $\partial S = \emptyset$.

20 Feburary 28th

Midterms covers:
Partial Fractions Decomposition
Before Reading week materials.
Bridges (Cut-edges)
A **bridge** in a graph $G = (V, E)$ is an edge $e \in E$ such that

$$c(G \setminus e) > c(G)$$

Here, $c(G)$ is the number of connected components of G.
Bridges
Conjectures:

- Bridges $\iff$ not in a cycle
- Bridge $\rightarrow c(G \setminus e) = 1 + c(G)$

Deleting a vertex can increase number of components arbitrarily.
Reduction to the connected case
Let G be a graph with components.

$$G_1, G_2, \ldots, G_c$$

Let $e \in E(G)$. Say $e \in (G_1)$.

- e is contained in a cycle of G iff e is contained in a cycle of G_1.
- e is a bridge of G iff e is a bridge of G_1.

Propositions:
Let $G = (V, E)$ be a graph and $e \in E$. Then e is a bridge iff e is not contained in any cycles of G.
Proof:
As above, we may assume that G is connected.
First, assume that e is in a cycle, C.
We want to show that $G \setminus e$ is connected.
G is connected, it has a vertex, $G \setminus e$ has same vertex-set, so $G \setminus e$ is not empty.
Let $u, v \in V$. Show that u reaches v in $G \setminus e$.

Since G is connected, there is a (u, v)-walk in G.
So there is a path P in G from u to v.
If P doesn't use e the P is a (u, v)-path in $G \setminus e$.
If P does use e, then it uses it once (since path has no repeated vertices).
$P : u - - - - - - - - - - xy - - - - - - - - - - -v$
Now, $C \setminus e = Q : x - y$ is a path in $G \setminus e$ from x to y.
Now $u - - - - - - - - -xQy - - - - - - - - - - - - v$ is a (u, v)-walk in $G \setminus e$.
So u reaches v in $G \setminus e$.
Therefore, $G \setminus e$ is connected.
Conversely, if e is not a bridge, then e is in a cycle.
Since $e = xy$ is not a bridge, $G \setminus e$ is connected.
So x reaches y in $G \setminus e$.
So there is an (x, y)-path P in $G \setminus e$.
Now, $(V(P), E(P) \cup \{e\})$ is a cycle in G containing e.

Proposition:

Let $G = (V, E)$ be a connected graph and $e = xy \in E$ a bridge. Then $G \setminus e$ has exactly two components X, Y with $x \in V(X)$ and $y \in V(Y)$.

Proof:

Let X be the component of $G \setminus e$ containing x.
Let Y be the component of $G \setminus e$ containing y.
Show $X \neq Y$ and $V(X) \cup V(Y) = V(G)$.
If $X = Y$, then there is (x, y)-path P in $G \setminus e$.
Now $P \cup \{e\}$ is a cycle of G containing e.
Previous proposition $Rightarrow$ e is not a bridge. Contradiction!
Thus, $X \neq Y$.
Consider any $z \in V(G)$.
There is a path, P, from x to z in G, since G is connected.
If P doesn't use e, then P is in $G \setminus e$, so $z \in V(X)$.
Since P has no repeated vertices, e is the first edge of the path $P : xy.......z$.
Now, we have a $(y - z)$-path in $G \setminus e$.
So $z \in V(Y)$.

21 March 2nd

Trees.

Midterms. 90 Enumeration.

A graph $G = (V, E)$ is minimally connected if G is connected and for every $e \in E$, $G \setminus e$ is not connected.

G is connected and every edge is a bridge.

Proposition:

G is minimally connected if and only if G is connected and contains no cycles.

Proof (Exercise):

A graph $G = (V, E)$ is a **tree** if it is connected and contains no cycles.

Small Trees

See pictures.

A **leaf** is a vertex of degree 1.

Proposition:

A tree T with at least two vertices has at least two leaves.

Proof:

Let P be a longest path in T.

$P : (v_0 v_1 \dots v_k)$. Then $l(P) \geq 1$ since $|V(T)| \geq 2$. and T is connected.

Now, both v_0 and v_k must be leaves.

Pictures.

Proposition:

A graph $G = (V, E)$ is a tree if and only if it is nonempty and for all vertices $v, w \in V$, there is exactly one (v, w)-path in G.

Proof:

First assume that G is a tree. Let $v, w \in V$.

Since G is connected, there is a (v, w)-path in G.

If there were ≥ 2 (v, w)-paths in G, then G would contain a cycle (by a previous Proposition).

Since G is a tree, this does not happen.

Second, assume that G is (nonempty and) not a tree.

So either G has $c(G) \geq 2$ components, or G contain a cycle.

If $c(G) \geq 2$, then let $v, w \in V$ be in different components of G. There is no (v, w)-path in G.

If $C = (v_0 v_1 v_2 \dots v_k v_0)$ is a cycle in G.

Then, $v_0 \neq v_k$ and $(v_0 v_1 \dots v_k)$ and $(v_0 v_k)$ are two different paths from v_0 to v_k in G.

A graph is a **forest** if it does not contain any cycles.

Any connected component of a forest is a tree.

Theorem:

Let $G = (V, E)$ be a graph with $|V| = n$ vertices, $|E| = m$ edges, and $c(G) = c$ components. Then,

$$m \geq n - c$$

with equality if and only if G is a forest.

Proof:

By induction on $|E| = m$.

Basis: $m = 0. E = \emptyset$.

So G has n vertices, 0 edges, $c = n$ components.

$0 \geq n - n$. Equality holds and K_n^c is a forest.

Induction hypothesis

If G' is a graph with $|V'| = n', |E'| = m', c(G') = c'$ components and $m' < m$ then $m' \geq n' - c'$ and equality holds iff G' is a forest.

Induction Step

G is as in the statement with $|E| = m \geq 1$.

Let $e \in E$ be an edge of G.

Now e is either a bridge in G or it is not.

Let $G' = G \setminus e$

$n' = n, m' = m - 1, c' = \begin{cases} c & \text{if } e \text{ is not a bridge} \\ c+1 & \text{if } e \text{ is a bridge} \end{cases}$

In either case, G' satisfies $m' \geq n' - c'$ with equality if and only if G' is a forest.

Case 1: e is a bridge.

Now $m = m' + 1 \geq (n' - c') + 1 = n' - (c' - 1) = n - c$

Proving the desired inequality.

Notice that $m = n - c$ if and only if $m' = n' - c'$

By induction, this happens if and only if G' is a forest.

Lemma:

Let H be a graph and $e \in E(H)$ a bridge.

Then H is a forest if and only if $H \setminus e$ is a forest.

Proof: (Exercise).

Case 2:

e is not a bridge.

Now $m = m' + 1 \geq (n' - c') + 1 = (n - c) + 1 > n - c$

Proving the desired inequality strictly.

Since e is a beidge in G, e is in a cycle of G, so G is not a forest.

Corollary:

$c = 1$ case of the Theorem.

A graph G is a tree iff it is connected and has $|E| = |V| - 1$.

22 March 4th

Corollary

If $G = (V, E)$ is a connected graph with $|V| = n$ vertices and $|E| = m$ edges, then $m \geq n - 1$, with equality if and only if G is a tree.

Numerology of Trees

Let $T = (V, E)$ be a tree with n vertices, $m = n - 1$ edges.

Let n_d be the number of vertices of degree d, for each $d \in \mathbb{N}$.

$$|V| = n = n_0 + n_1 + n_2 + n_3 + \dots$$

By the Handshake Lemma,

$$2|E| = n_1 + 2n_2 + 3n_3 + \dots$$

Since

$$2|V| = 2 + 2|E|$$

(Since T is a tree)

$$2(n_0 + n_1 + n_2 + n_3 + \dots) = 2 + n_1 + 2n_2 + 3n_3 + \dots$$

$$2n_0 + n_1 = 2 + n_3 + 2n_4 + 3n_5 + \dots$$

If $n = 1$, then $n_0 = 1$ and $n_d = 0$ for all $d \geq 1$.

If $n \geq 2$, then $n_0 = 0$ since T is connected.
So for a tree, T with $n \geq 2$ vertices,

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots \geq 2$$

Spanning Trees
Let $G = (V, E)$ be a graph. A spanning tree of G is a subgraph. $T = (V, F)$ that is

- spanning
- and a tree

Proposition:
Let $G = (V, E)$ be a graph. Then G has a spanning tree if and only if G is connected.
Proof:
If G has a spanning tree, then G is connected, since T is a connected spanning subgraph of G.
Conversely, we go by induction on $|E|$. Fix $|V| = n$.
Basis of induction:
$|E| = n - 1$. Then G is a tree.
So G is a spanning tree of itself.
Induction Step:
Let G be connected with $|V| = n$ vertices and $|E| > n - 1$ edges.
So G is connected but not a tree.
So G contains a cycle C.
Let e be an edge of C.
So e is not a bridge of G.
So $G \setminus \{e\}$ is connected, with $|E(G \setminus \{e\}| = |E| - 1)$.
By induction, $G \setminus \{e\}$ has a spanning tree T.
Since $G \setminus \{e\}$ is a spanning subgraph of G, T is also a spanning tree of G.
Theorem:
A graph $G = (V, E)$ is bipartite if and only if it does not contain an odd cycle.
Proof:
We have seen that if G contains an odd cycle, then G is not bipartite.
Claim:
We can reduce to the case that G is connected.
So assume that G is connected and not bipartite.
Since G is connected, it contains a spanning tree T.
G.
Lemma:
Since T is a tree, it has a bipartition (A, B).
Proof: (Exercise)
(Induct on $|V(T)|$ by deleting a leaf.)

Since G is not bipartite, there is an edge $e = vw \in E$ of G with both ends on the same side - both ends in A, say.

Since T is a tree, there is a unique path P in T from v to w.

Since (A, B) is a bipartition of T and both $v, w \in A$, the path P has an even number of edges.

Now $(V(P), E(P) \cup \{vw\})$ is an odd cycle in G.

Two-out-of-Three Theorem

Let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges.

Consider the following three properties.

1. G is connected.
2. G has no cycles.
3. $m = n - 1$.

Any two of these properties imply the other one.

Proof:

(1) and (2) imply (3).

Assume that G is connected and has no cycles.

So $m = n - 1$ by the Corollary at the start of today.

(1) and (3) imply (2)

Assume that G is connected and $m = n - 1$.

So G is a tree by the Corollary at the start of today.

(2) and (3) imply (1)

Look at each connected component of G.

23 May 6th

Let G satisfy (2) and (3).

Let $G_1, G_2, \ldots, G_c$ be the connected components of G.

Each connected component G satisfies (1) and (2).

So G_i has n_i vertices and m_i edges and $m_i = n_i - 1$ (Since we know that (1) and (2) $\Rightarrow$ (3).

Now, since G satisfies (3), $1 = n - m = (n_1 + n_2 + \cdots + n_c) - (m_1 + m_2 + \cdots + m_c)$ $= (n_1 = m_1) + (n_2 - m_2) + \cdots + (n_c - m_c) = c$

So G is connected.

Search Trees

Is there a walk (or a path) in G from v_* to z?

Algorithm

Input graph $G = (V, E)$ and "root" vertex $v_* \in V$.

Let $W = \{v_*\}$ and let $F = \emptyset$.

Let $\text{pr}(v_*) = null$ and $l(v_*) = 0$.

Let $\Delta = \partial W = \{e \in E : |e \cap W| = 1\}$

while $\Delta \neq \emptyset$

Pick any $e = xy \in \Delta$ with $x \in W$ and $y \notin W$.

Update $F := F \cup \{e\}$ and $W := W \cup \{y\}$

pr$(y) := x$ and $l(y) = 1 + l(x)$.
Recalculate $\Delta = \partial W$.
Output:
$T = (W, F)$ and pr: $W \to W \cup \{null\}$ and $l : W \to \mathbb{N}$.
Picture here.
Theorem:
With the above notation.

1. $T = (W, F)$ is a spanning tree for the component of G that contains v_*.
2. For all $w \in W$, the unique path from w to v_* in T is obtained by following the steps $v \to \text{pr}v$ until $\text{pr}(v) = \text{null}$.
3. The length of this path is $l(w)$.

Proof:
Claim: $T = (W, F)$ is a tree and (2) and (3) holds.
By induction on the number of iterations of the "while ($\Delta \neq \emptyset$)" loop.
Basis: $W = \{v_*\}, F = \emptyset$, and $T = (\{v_*\}, \emptyset)$ is a tree. pr and l are defined on W and (2) and (3) hold.
Consider $e = xy \in \Delta$ with $x \in W$ and $y \notin W$.
Let W', F', pr', l' be the updated data.
By induction, (W, F) is a tree. Connected and $|F| = |W| - 1$.
Now, (W', F') is connected and $|F'| = |W'| - 1$.
By 2-out-of-3 THeorem, $T' = (W', F')$ is a tree.
Check (b) and (c) for y.
Claim: T is a spanning tree for the component of G that contains v_*.
Show: If v_* reaches $z \in V$ in G, then $z \in W$.
Suppose not.
Suppose $z \in V$ is such that v_* reaches z but $z \notin W$.
Let Z be a walk from v_* to z in G.
$v_* \in W$ and $z \notin W$.
There is a step xy of z with $x \in W$ and $y \notin W$.
But now $xy \in \Delta$, contradicting $\Delta = \emptyset$ because the algorithm terminated.
Application:

1. Finding components.
2. Finding paths between vertices (in a connected graph).
3. Finding cycles.
4. Testing bipartiteness

24 March 9th

Planar Graphs

Which graphs can be drawn in the plane $\mathbb{R}^2$ without crossing edges?
$\mathcal{P} = \{p_v : v \in V\}$ distinct points in $\mathbb{R}^2$ representing vertices.
$\Gamma = \{\gamma_e : e \in E\}$ distinct (simple) curves in $\mathbb{R}^2$ representing edges.

- If $e = xy$, then γ_e has endpoints p_x and p_y.
- Edges don't cross.
- Other conditions.

Small examples
Complete graphs.
See picture.
Complete Bipartite Graphs.
See picture.
A graph G is **planar** if it has a plane embedding.
Lemma:
Every subgraph of a planar graph is planar.
Subdivision
Let $G = (V, E)$ be a planar graph, $e = xy \in E$, and $z \notin V$.
The subdivision of e in G is $G \cdot e$.
Vertex-set

$$V(G \cdot e) = V(G) \cup \{z\}$$

Edge-set

$$E(G \cdot e) = (E(G) \setminus \{e\}) \cup \{xz, yz\}$$

Repeated subdivision: do this 0 or more times.
Lemma:
$G = (V, E)$ is planar if and only if $G \cdot e$ is planar.
Shape of the Proof:
First, assume that G is planar.
Let (P, Γ) be a plane embedding of G.
Construct a plane embedding of $G \cdot e$.
Let $\gamma_e : [0, 1] \to \mathbb{R}^2$ be the simple curve $\gamma_e \in \Gamma$ representing e.
(γ_e is a continuous (tame) **injective** function)
Let $p_z = \gamma_e\left(\frac{1}{2}\right)$.
Define:
$\gamma_{xz} : [0, 1] \to \mathbb{R}^2$ by $\gamma_{xz}(t) = \gamma_e(\frac{t}{2})$
$\gamma_{xz}(0) = \gamma_e(0) = p_x$ $\gamma_{xz}(1) = \gamma_e(\frac{1}{2}) = p_z$
Similarly, $\gamma_{yz} : [0, 1] \to \mathbb{R}^2$, $\gamma_{yz}(t) = \gamma_e(1 - \frac{t}{2}), \gamma_{yz}(0) = \gamma_e(1) = p_y, \gamma_{yz}(1) = \gamma_e(\frac{1}{2}) = p_z$
Check:
This gives a planar embedding of $G \cdot e$.

Converse:
Given a plane embedding of $\Gamma \cdot e$. Construct a plane embedding of G.
Conjecture
If G contains a (repeated) subdivision of K_5 or $K_{3,3}$, then G is not planar.
Proof:
(Wednesday: K_5 and $K_{3,3}$ are not planar.)
Kuratowski's Theorem (1930)
A graph is planar if and only if it does not contained a subdivision of K_5 or $K_{3,3}$ as a subgraph.
"Kuratowski subgraphs".

25 March 11th

Last time: K_5 and $K_{3,3}$ are non-planar.
Faces of plane embedding
Let $G = (V, E)$ be a plane embedded graph and let F be a face of G. The boundary of F is the subgraph of G consisting of the vertices and vertices incident to F.

The **degree of** F is the number of edges in the boundary plus the number of bridges in the boundary.

Lemma: An edge e of a planar embedded graph G is a bridge iff the faces on either side of e are the same face.
Theorem:
Let $G = (V, E)$ be a planar embedded graph and let F be the set of faces. Then

$$\sum_{F \in \mathcal{F}} \deg(F) = 2|E|$$

The faceshaking Lemma (FSL)
Proof:
When we sum the degrees of the faces, we are counting every edge twice.
Theorem: (Euler's Formula)
Let $G = (V, E)$ be a planar embedded graph with n vertices, m edges, f faces and c components.
Then

$$n - m + f = c + 1$$

Proof:
We proceed by induction on m. If $m = 0$, then $f = 1$ and $c = n$.
Let $m \geq 1$, suppose that the formula holds for plane embedded graphs with fewer than m edges. Let $e \in E$ and consider $G' = G \setminus e$. G' has n vertices, $m - 1$ edges.
Let f' be the number of faces and c' be the number of components. Then

$$n - (m - 1) + f' = c' + 1 \qquad (*)$$

If e is a bridge, then $c' = c + 1$ and $f' = f$

Then $(*)$ gives $n - m + 1 + f = c + 2$.
If e is not a bridge, then $c' = c$ and

$$f' = f - 1$$

, so $(*)$ gives $n - m + 1 + f - 1 = c + 1$.
Theorem:
Let $G = (V, E)$ be a connected planar graph with $n \geq 3$ vertices and m edges. Then $m \geq 3n - 6$ and equality holds iff every face in every plane embedding of G has degree 3.
Proof:
Let $\mathcal{F}$ be the set of faces in a plane embedding of G, and let $f = |\mathcal{F}|$.
Since $n \geq 3$, every face has degree at least 3.
Therefore,

$$2m = \sum_{F \in \mathcal{F}} \deg(F) \geq 3f \Rightarrow f \geq \frac{2m}{3}$$

By Euler's Formula,

$$n - m + f = 2$$

$$\Rightarrow m = n + f - 2 \leq n + \frac{2m}{3} - 2$$

...
$m \leq 3n - 6$.
(Proof by Exercise)
Equality holds iff $2m = 3f$ iff every face has degree 3.
Claim: K_5 is non-planar.
Proof: K_5 is connected with 5 vertices, and $\binom{5}{2} = 10$ edges. $3 \cdot 5 - 6 = 9 < 10$.

26 March 13th

Numerology of PLanar Graphs
Let $G = (V, E)$ be a graph with a plane embedding (P, Γ).
$|V| = n$ vertices, $|E| = m$ edges, $|\mathcal{F}| = f$ faces, $c(G) = c$ components.

1. Handshake: $2m = \sum_{v \in V} \deg(v)$
2. Faceshaking: $2m = \sum_{F \in \mathcal{F}} \deg(F)$
3. Euler's Formula: $m - n + f = c + 1$

Lemma: If G has at least two edges, then every face of every plane embedding of G has degree at least 3.
Proof:
Induct on $|E|$. Exercise.
If G is connected and $|V| \geq 3$, then $|E| \geq 2$.

Corollary: If G is planar and connected, then $|E| \leq 3|V| - 6$ with equality iff every face of every embedding of G has degree 3.

Proof:

Consider any plane embedding of G.

$$2m = \sum_{F \in \mathcal{F}} \deg(F) \geq 3f$$

by the Lemma.

Multiply Euler's Formula by 3:

$$6 = 3(c+1) = 3n - 3m + 3f \leq 3n - 3m + 2m = 3n - m$$

So $m \leq 3n - 6$.

Corollary: K_5 is not planar.

$$|E| = 10 \not\leq 9 = 3|V| - 6$$

Corollary: If G is connected, planar, $|V| \geq 3$, with no 3-cycles.

Then $|E| \leq 2|V| - 4$ with equality if and only if every face of every plane embedding of G has degree 4.

Proof:

Consider any plane embedding. All faces have degree at least 4.

Faceshaking lemma: $2m \geq 4f$.

Multiply Euler's Formula by 4: $8 = 4(c+1) = 4n - 4m + 4f \leq 4n - 2m$

So $m \leq 2n - 4$, statement about equality also follows.

Example: $K_{3,3}$ is not planar.

$$|E| = 9 \not\leq 8 = 2 \cdot |6| - 4$$

Let (P, Γ) be a plane embedding of a graph $G = (V, E)$.

$|V| = n \geq 3, |E| = m, |\mathcal{F}| = f, c(G) = 1$

Say there are n_d vertices of degree $d \in \mathbb{N}$.

$n_0 = 0$ since G is connected and $|V| \geq 3$.

Euler's Formula:

$$n + m - f = 2$$

$$n = n_1 + n_2 + n_3 + n_4 + \dots$$

$$2m = n_1 + 2n_2 + 3n_3 + 4n_4 + \dots$$

Since every face has degree ≥ 3:

$$2m = \sum_{F \in \mathcal{F}} \deg(F) \geq 3f$$

Multiply Euler's Formula by 6.

(Equality iff every face has degree 3).

$$12 = 6(c+1) = 6n + 6m - 6f \leq 6n - 2m$$

$$12 \leq 6(n_1 + n_2 + \dots) - (n_1 + 2n_2 + 3n_3 + \dots)$$

$$5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 \geq 12 + n_7 + 2n_8 + 3n_9 + \dots$$

Equality iff every face of every planar embedding has degree 3.

Exercise. What is the analogue if G has no 3-cycles?

Exercise: What is the analogue counting faces of degree d instead of vertices of degree d?

(Require all vertices to have degree $\geq k$).

Corollary: If G is a connected planar graph, then G has a vertex of degree at most 5.

Platonic Solids

Let $k \geq 3$ and $d \geq 3$.

When is there a (finite) plane embedding in which

- G is connected
- Every vertex has degree d.
- Every face has degree k.

We have

- Handshake: $dn = 2m$, So $n = 2m/d$
- Faceshaking: $fk = 2m$, So $f = 2m/k$
- Euler's Formula: $n + m - f = 2$,

$$\frac{2m}{d} + \frac{2m}{k} = 2 + m$$

So

$$\frac{1}{d} + \frac{1}{k} = \frac{1}{2} + \frac{1}{m} > \frac{1}{2}$$

See pictures.